Refined Quorum Systems

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Abstract. It is considered good distributed computing practice to devise object implementations that tolerate contention, periods of asynchrony and a large number of failures, but perform fast if few failures occur, the system is synchronous and there is no contention. This paper initiates the first study of quorum systems that help design such implementations by encompassing, at the same time, optimal resilience, as well as *optimal best-case complexity.*

We introduce the notion of a *refined* quorum system (RQS) of some set S as a set of three classes of subsets (quorums) of S: first class quorums are also second class quorums, themselves being also third class quorums. First class quorums have large intersections with all other quorums, second class quorums typically have smaller intersections with those of the third class, the latter simply correspond to traditional quorums. Intuitively, under uncontended and synchronous conditions, a distributed object implementation would expedite an operation if a quorum of the first class is accessed, then degrade gracefully depending on whether a quorum of the second or the third class is accessed. Our notion of refined quorum system is devised assuming a general adversary structure, and this basically allows algorithms relying on refined quorum systems to relax the assumption of independent process failures, often questioned in practice.

We illustrate the power of refined quorums by introducing two new optimal Byzantine-resilient distributed object implementations: an atomic storage and a consensus algorithm. Both match previously established resilience and best-case complexity lower bounds, closing open gaps, as well as new complexity bounds we establish here. Each of our algorithms is representative of a different class of architectures, highlighting the generality of the refined quorum abstraction.

Keywords: Quorums, Consensus, Complexity, Shared-memory emulations, Byzantine failures.

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* Elements of this paper appeared in a paper with the same title in the Proceedings of the 26th ACM Symposium on Principles of Distributed Computing (PODC'07).

1 Introduction

1.1 Background

Quorum systems are powerful mathematical tools to reason about distributed implementations shared objects, in particular read/write storage (e.g., [4, 27, 40]) and consensus [8, 13, 34] abstractions. More specifically, quorum systems have been used (either explicitly or implicitly) to reason about distributed algorithms that tolerate process failures, as well as arbitrarily long periods of asynchrony, also called indulgent algorithms [19]. Originally, a quorum system was defined as a set of subsets that intersect [16], and this notion was key to reasoning about crash-resilient asynchronous algorithms. More sophisticated forms of quorum systems have been introduced to cope with Byzantine (malicious) failures [37]: these require larger intersections among subsets [40].

However, while being very useful to reason about the resilience dimension, traditional quorums (be they simple or Byzantine) are not adequate to capture the complexity dimension. This is particularly important given the appealing nature of *optimistic* distributed object implementations, e.g., [1,7,10,12,18,20,31,36,41,46,52]. In addition to being indulgent, these implementations are also geared to reduce best-case complexity, i.e., latency under situations of synchrony and no-contention, which are typically argued to be frequent in practice. More specifically, these implementations are tuned to expedite operations in uncontended and synchronous situations, provided *"enough"* servers are accessed. This very notion of *"enough servers to expedite an operation"* is crucial, but is not captured by traditional quorum systems. It is natural to seek for a mathematical abstraction that captures it in precise yet general terms. This was the motivation of this work.

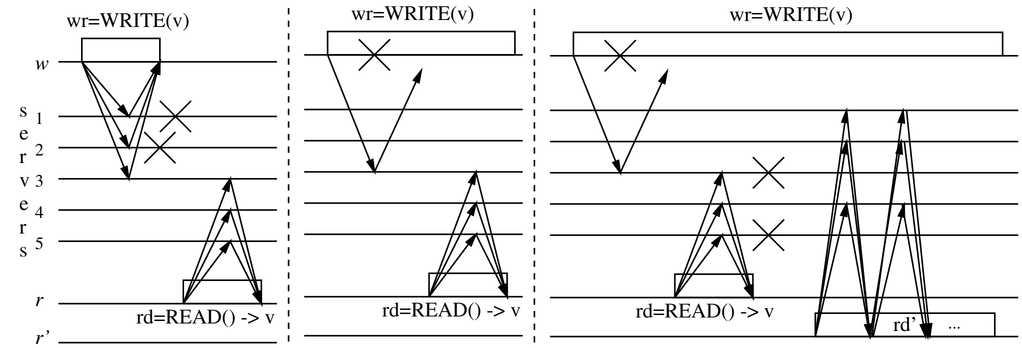
1.2 Example

To illustrate the motivation, consider the simple context of a crash-resilient asynchronous implementation of a wait-free atomic storage over a set of server processes [4]. It is known [11] that no optimally resilient atomic storage algorithm can have both reads and writes complete in a single communication round-trip (we simply say round), even if a single writer is involved (SWMR). For instance, the classical, optimally crash-resilient solution [4] (that assumes a majority of correct processes) requires two rounds for a read.

As we discussed earlier, it is practically appealing to look into best-case complexity and ask if it is possible to expedite *both* reads and writes within a single round in a synchronous and contention-free period. Clearly, if the reader (resp. the writer) access all servers in the first round, then it can immediately return a valid response. But do we need to access all servers for that? How many servers actually *need* to be accessed to achieve such a *fast termination* in best-case conditions?

Consider $S = 5$ servers implementing a crash-tolerant atomic wait-free storage assuming $t = 2$ server failures (optimal resilience). We argue below that any algorithm that greedily expedites read/write operations in one round during synchronous and contention-free periods whenever $S-t = 3$ servers are accessed, violates atomicity. This is depicted through several executions of such an algorithm (Figure 1):

1. In the first execution ($ex1$), writer w invokes $wr = \mathsf{write}(v)$ and servers 4 and 5 are faulty. Then, wr writes value v into the subset of servers $Q_1 = \{1, 2, 3\}$ and completes in a single round.
2. The second execution ($ex2$, Fig. 1(a)) is slightly different because servers 4 and 5 are actually correct. Yet wr also completes in a single round, after writing in Q_1. Then servers 1 and 2 crash

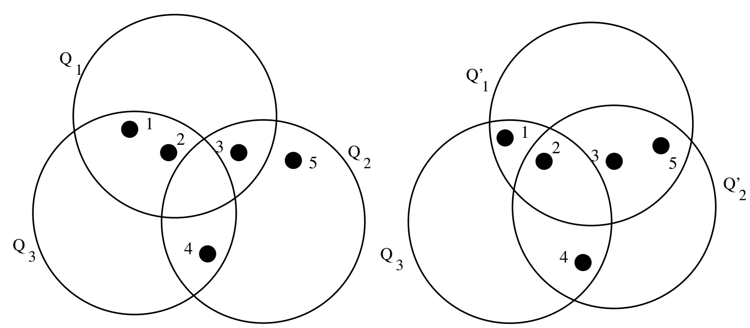


(a) ex2 (b) ex3 (c) ex4

Fig. 1. Violation of atomicity in case the single-round operations access only 3 servers.

and a read rd (by the reader r) is invoked. Assuming synchrony and no contention, rd accesses server set $Q_2 = \{3, 4, 5\}$ and completes in a single round.

3. The third execution ($ex3$, Fig. 1(b)) is similar to $ex2$ except that (1) the write is incomplete and writes only to server 3, (2) servers 1 and 2 (i.e., servers from the set $Q_2 \setminus Q_1$) are correct, but the communication between the reader and servers from $Q_2 \setminus Q_1$ is delayed. Read rd does not distinguish $ex3$ from $ex2$ and completes in a single round, returning v.
4. Finally, the fourth execution ($ex4$, Fig. 1(c)) extends $ex3$ by: (1) the crash of servers 3 and 5 and (2) the invocation of read rd' by a different reader r'. This reader cannot return v using $Q_3 = \{1, 2, 4\}$ regardless of how many rounds are used. Atomicity is violated.



(a) $|Q_1| = |Q_2| = 3$ (b) $|Q_1'| = |Q_2'| = 4$

Fig. 2. Quorum intersections

Essentially, atomicity is violated because $Q1 \cap Q2 \cap Q3 = \emptyset$ (Figure 2(a)). On the other hand, we can devise a storage algorithm that achieves fast termination whenever 4 servers are accessed.

For instance, consider the following algorithm, a variation of [4], in which all servers maintain 2 timestamp/value variables pw and w:

– On invoking wr =write(v), as in [4], the writer increments its local timestamp ts and assigns it to value v and writes the pair to servers' pw variable. However, unlike in [4], the write completes in a single round only if it writes $\langle ts, v\rangle$ to 4 servers, say $Q_1' = \{1, 2, 3, 5\}$. Otherwise, if the writer reaches only 3 servers in the first round (after waiting for some pre-specified time, complying with synchrony assumptions), the writer invokes the second round of write and writes $\langle ts, v\rangle$ to servers' w variable. The write completes when the writer receives *acks* in the second round from any 3 servers.

- On the other hand, the first round of read, similarly as in [4], collects the servers' local copies pw and w. The reader selects a timestamp/value pair $c_{max} = \langle ts_{max}, v\rangle$ with the highest timestamp. Unlike in [4], the read completes (and returns v) at the end of the first round, if c_{max} is read in 3 different pw fields, or in some w field. It is very important to notice that this is always the case if there is no read/write contention and the reader receives a response from 4 (or more) servers in the first round, say $Q'_2 = \{2, 3, 4, 5\}$. Intuitively, the read may safely return after the first round if it makes sure that it leaves behind at least 3 servers with the knowledge of the latest value. This is the case both if the read accesses 3 servers from Q'_1 (e.g., $Q'_1 \cap Q'_2 = \{2, 3, 5\}$) that report the latest value in their pw variables, or if the reader selects the highest value from a server w variable, meaning that the writer already informed 3 servers about the latest value.
- Otherwise, if there is contention, or only 3 servers are available, the reader may not be able to return the value at the end of the first round. In this case, after reaching 3 servers in the first round, the reader proceeds to the second round, in which, as in [4], the reader writes back $c_{max} = \langle ts_{max}, v\rangle$ to servers' pw field. The read completes when the reader receives *acks* from any 3 servers in the second round.

In the above example, the key to ensuring atomicity while allowing both reads and writes to terminate in a single round is to have $Q'_1 \cap Q'_2 \cap Q_3 \neq \emptyset$, where Q_3 is any quorum in the system (in our case any subset of 3 or more servers). Namely (Figure 2(b)), in a system of 5 elements, any two subsets of 4 elements intersect with any subset of 3 elements. Basically, boosting complexity requires to access subsets of servers that have larger intersections than traditional quorums. The above example is (relatively) simple because we considered: a) crash failures only, b) a threshold adversary (at most t faulty processes) and c) no graceful degradation (i.e., achieving the next best possible latencies, when the best possible one (e.g., a single round) cannot be achieved).

The motivation underlying this paper is precisely to characterize the required intersection properties in a precise and general manner. We aim at a characterization that is necessary and sufficient for optimizing the best-case complexity of various distributed object implementations, in various failure models, under various adversary structures, and also considering graceful degradation.

1.3 Contributions

This paper introduces the notion of *refined quorum systems (RQS)*. In short, a refined quorum system of some set of elements S is a set of three classes of subsets (quorums) of S: first class quorums are also second class quorums, which are also third class quorums. Quorums of the first class have large intersections with quorums of other classes, those of the second class typically have smaller intersections with those of the third class, the latter simply correspond to traditional quorums. In the context of a distributed object implementation, the set S would typically contain the set of fault-prone server processes over which some object abstraction (e.g., storage or consensus) is implemented.

Intuitively, under uncontended and synchronous conditions, a distributed object implementation would expedite an operation if a quorum of the first class is available, then degrade gracefully depending on whether a quorum of the second or the third class is available. We argue that our quorum notion is, in a sense, complete: there is no need for further refinement of quorums with the goal of optimizing best-case efficiency. Indeed, the properties provided by our third class quorums are anyway necessary for hindering the partitioning of the asynchronous system, which is key to any resilient distributed atomic storage or consensus implementation. Hence, there is no need to consider weaker intersection properties. Moreover, and as we show in this paper, optimally resilient and best-case efficient implementations of the seminal register and consensus abstractions have exactly *three* possible latencies under uncontended and synchronous conditions. This observation is of independent interest.

Our refined quorum systems are designed to handle a general adversary structure in which various subsets of processes can collude to defeat the protocol [26, 29, 40]. With such a general structure, we relax the often criticized assumption, of independent and identically distributed failures [1, 7, 10, 12, 18, 20, 36, 41, 46, 52].

We illustrate the power of our notion of refined quorum systems by introducing two new atomic object implementations. Each algorithm is interesting in its own right and is, in a precise sense, the first fully optimal protocol of its kind in terms of best case complexity.

- Our first object implementation is a new Byzantine-resilient asynchronous distributed storage algorithm implementing the atomic register abstraction. Such algorithms constitute an active area of research and are appealing alternatives to classical centralized storage systems based on specialized hardware [48]. The challenge when devising distributed storage algorithms is to ensure that *reads* and *writes* have low latency in most frequent situations, while (a) tolerating asynchrony and the failures of a large number of base servers (typically commodity disks) as well as any number of clients that access the storage (wait-freedom [24]) and (b) ensuring strong consistency (ideally atomicity [25, 33]). Using a refined quorum system, we present an atomic wait-free storage algorithm that combines optimal resilience with the lowest possible *read/write* latency in best-case conditions (no-contention and synchrony). Under such conditions, our algorithm expedites storage operations (*reads* and *writes*) in a single communication round-trip (or simply, round) if a first class quorum is accessed, in two rounds if a second class quorum is accessed and in three rounds otherwise. The latter case is when a third class quorum is available which is a necessary condition for resilience anyway. Our algorithm does not rely on any data authentication primitive, and matches the resilience and complexity lower bounds of [20, 42] (even when these bounds are extended to a general adversary structure), together with a new bound we establish in this paper. Our new bound captures the best-case complexity of gracefully degrading atomic storage implementations.
- Our second algorithm implements a Byzantine-resilient consensus abstraction in the general state machine replication (SMR) framework of [34], distinguishing different process roles: *proposers* that propose values to be learned by *learners* with the mediation of *acceptors*. Our algorithm is the first to tolerate (1) any number of Byzantine failures of proposers and learners, (2) the largest possible number of acceptor failures, and (3) arbitrarily long periods of asynchrony. On the other hand, under best-case conditions, our algorithm allows a value to be learned in only two message-delays in case a first class quorum is accessed, and in three (resp., four) message delays in case a second (resp., third) class quorum is accessed. Note here that (a) learning in a single message delay is obviously impossible with multiple or potentially Byzantine proposers, and (b) the availability of a third class quorum is anyway necessary for resilience. Our algorithm matches the resilience and complexity lower bounds of [35] (including when these bounds are extended to a general adversary structure), together with a new comple-

mentary bound we establish here on consensus algorithms that degrade gracefully in best-case executions. These bounds state minimal conditions under which the state-machine replication approach can be made optimally resilient and best-case efficient. Until now, it was not clear whether the conditions of [35] were also sufficient. We show they are and we complement them.

We believe that it would have been very hard to devise such algorithms, especially in the context of a general adversary structure, without the notion of a refined quorum system, though we might be subjective here.

1.4 Roadmap

The rest of the paper is organized as follows. Section 2 first presents our quorum notion and illustrates how it generalizes previous ones through examples from the literature. Sections 3 and 4 introduce our two new distributed object implementations that exploit the full features of refined quorums. Proofs of correctness of our two algorithms are postponed to the appendices for better readability of the paper. We complete the overview of related work in Section 5. Then, we conclude the paper by pointing out some open research directions.

2 Refined Quorum Systems

2.1 Definitions

A refined quorum system is expressed in the abstract context of a non-empty set S of elements, and an *adversary structure* (or, simply, *adversary*) $\boldsymbol{B}$ defined as follows [26]:

Definition 1. *Let $\boldsymbol{B}$ be any set of subsets of S. $\boldsymbol{B}$ is an* adversary *(for S) if:* $\forall B \in \boldsymbol{B}$: $B' \subseteq B \Rightarrow B' \in \boldsymbol{B}$.

Let $\boldsymbol{QC_1}$, $\boldsymbol{QC_2}$ and $\boldsymbol{RQS}$ be any set of subsets of S, such that $\boldsymbol{QC_1} \subseteq \boldsymbol{QC_2} \subseteq \boldsymbol{RQS}$. We define our quorum notion through three properties on $\boldsymbol{QC_1}$, $\boldsymbol{QC_2}$ and $\boldsymbol{RQS}$. For every property, we first give a basic intuition and then the formal statement.

In the following, we refer to elements of $\boldsymbol{QC_i}$ as *class i elements*. We also sometimes write $\boldsymbol{QC_3} = \boldsymbol{RQS}$, and refer to an element of $\boldsymbol{RQS}$ that is not a class 2 element as a *class 3* element.

Informally, Property 1 states that the adversary must not control an intersection of any two elements of $\boldsymbol{RQS}$. Intuitively, the adversary could otherwise cause partitions in the system.

Property 1. The intersection of any two elements of $\boldsymbol{RQS}$ does not belong to $\boldsymbol{B}$, i.e.,
$P1(\boldsymbol{RQS},\boldsymbol{B}) \equiv \forall Q, Q' \in \boldsymbol{RQS}$: $Q \cap Q' \notin \boldsymbol{B}$.

Property 2 states that no two elements of the adversary structure may "cover" the intersection of two class 1 elements and some (class 3) element.

Intuitively, this is motivated by a latency consideration: one can think of two "lucky" clients (e.g., a writer and a reader), each operating on a class 1 quorums and achieving optimal latency (e.g., a single round), which does not allow for a written/read value to be "confirmed". If 2 elements of the adversary structure "cover" the above mentioned intersection, the adversary may leave the information about past operations only in one of its two elements, the other simply "forgetting" about the two "lucky" clients. This forces the third client (e.g., a reader) to find the information about the past operations only at servers (elements of S) that all belong to the set of servers that can be simultaneously controlled by the adversary. However, the third client cannot trust this information: since it was not "confirmed", the adversary might be simply forging it.

Property 2. The intersection of any two class 1 elements and any element of $\boldsymbol{RQS}$ is not a subset of the union of any two elements of $\boldsymbol{B}$, i.e.,

$P2(\boldsymbol{QC_1},\boldsymbol{RQS},\boldsymbol{B}) \equiv \forall Q_1, Q_1' \in \boldsymbol{QC_1}$, $\forall Q \in \boldsymbol{RQS}$, $\forall B_1, B_2 \in \boldsymbol{B}$: $Q_1 \cap Q_1' \cap Q \nsubseteq B_1 \cup B_2$.

Property 3 is slightly more involved than Property 2. It relates an intersection X of a class 2 element and a class 3 element with a given element of adversary structure B.

Informally, Property 3 states that, for any B: (a) X is not "covered" by B and some other element of the adversary structure, *or* (b) the intersection of X with each class 1 element is not "covered" solely by B. For example, in a distributed atomic storage context, Property 3 is crucial to facilitating both single round operations and graceful degradation to 2-round operations. The basic intuition behind this interesting property is that if a client accesses a class 2 element, the distributed service can respond somewhat in a slower manner (compared to the case when class 1 element is accessed), hence allowing for *some* "confirmation" of the clients' operations. This allows for intersections mandated by Property 3 to be smaller than those of Property 2 (yet larger than those of Property 1). In addition, there is also an interesting interplay between all 3 classes of elements. We briefly postpone a more detailed intuition of Property 3 to Example 7 in Section 2.2.

Property 3. Let X be an intersection of any class 2 element Q_2 and any element Q of $\boldsymbol{RQS}$, and let B be any element of $\boldsymbol{B}$. Then:
(a) the set difference between X and B does not belong to $\boldsymbol{B}$ (we say $P_{3a}(Q_2, Q, B)$ holds), *or*
(b) an intersection of any class 1 element[1] and X is not a subset of B ($P_{3b}(Q_2, Q, B)$ holds), i.e.,

$P3(\boldsymbol{QC_1},\boldsymbol{QC_2},\boldsymbol{RQS},\boldsymbol{B}) \equiv \forall Q_2 \in \boldsymbol{QC_2}$, $\forall Q \in \boldsymbol{RQS}$, $\forall B \in \boldsymbol{B}$:

$$(Q_2 \cap Q \setminus B \notin \boldsymbol{B}) \vee (\boldsymbol{QC_1} \neq \emptyset \wedge \forall Q_1 \in \boldsymbol{QC_1}\colon Q_1 \cap Q_2 \cap Q \nsubseteq B).$$

We are now ready to define a *refined quorum system.*

Definition 2. ***Refined Quorum System.*** *We say that* $\boldsymbol{RQS}$ *is a* refined quorum system *for a set* S *and adversary* $\boldsymbol{B}$, *if* $\boldsymbol{RQS}$ *has two subsets* $\boldsymbol{QC_1} \subseteq \boldsymbol{QC_2} \subseteq \boldsymbol{RQS}$ *such that properties* $P1(\boldsymbol{RQS},\boldsymbol{B})$, $P2(\boldsymbol{QC_1},\boldsymbol{RQS},\boldsymbol{B})$ *and* $P3(\boldsymbol{QC_1},\boldsymbol{QC_2},\boldsymbol{RQS},\boldsymbol{B})$ *hold.*

We simply call elements of a refined quorum system — *quorums.* Note that class 1 quorums are also class 2 quorums, which are also class 3 quorums. Notice also that, when $\boldsymbol{QC_1} = \boldsymbol{QC_2}$, Property 2 implies Property 3. Furthermore, when $\boldsymbol{B} = \emptyset$, Property 1 implies Property 3. Therefore, Property 3 is interesting on its own only if $\boldsymbol{B} \neq \emptyset$ and $\boldsymbol{QC_1} \neq \boldsymbol{QC_2}$.

In the following, we give illustrations of our quorum notion and explain how it extends traditional ones. Later in the paper, we will introduce new optimal algorithms that make full use of our quorum notion.

2.2 Examples

To get further intuition on RQS properties, we instantiate them here in the context of a *k-bounded threshold adversary*, denoted $\boldsymbol{B^k}$. This is a special case of an adversary that contains all subsets of

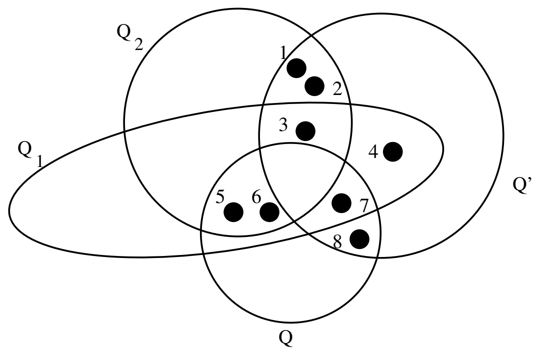
[1] Assuming there is at least one class 1 element, i.e., $\boldsymbol{QC_1} \neq \emptyset$.

S with cardinality at most k (i.e., $\boldsymbol{B^k} = \{B | B \subseteq S \wedge |B| \leq k\}$). In this context, the RQS properties can be expressed as follows:

Property 1. Any two quorums intersect in at least $k+1$ elements.

Property 2. The intersection of any two class 1 quorums intersects with any quorum in at least $2k+1$ elements.

Property 3. Any class 2 quorum intersects with any quorum in at least $2k+1$ elements or this intersection itself intersects with any class 1 quorum in at least $k+1$ elements.



$Q=\{5,6,7,8\}$

$Q'=\{1,2,3,4,7,8\}$

$Q_2=\{1,2,3,5,6\}$

$Q_1=\{3,4,5,6,7\}$

RQS= $\{Q,Q',Q_2,Q_1\}$
QC2= $\{Q_2,Q_1\}$
QC1= $\{Q_1\}$

$\cap$	Q	Q'	Q_2	Q_1
Q	Q	{7,8}	{5,6}	{5,6,7}
Q'	{7,8}	Q'	{1,2,3}	{3,4,7}
Q_2	{5,6}	{1,2,3}	Q_2	{3,5,6}
Q_1	{5,6,7}	{3,4,7}	{3,5,6}	Q_1

Fig. 3. Example of a RQS for an adversary $\boldsymbol{B^k}$ ($k=1$). Every pair of depicted sets intersects in at least $k+1$ elements (satisfying Property 1). Q_1 intersects with every other set in at least $2k+1$ elements (satisfying Property 2, for an intersection with itself). Moreover, for every $B \in \boldsymbol{B^k}$, $P_{3a}(Q_2, Q', B)$ and $P_{3a}(Q_2, Q_1, B)$ hold (since $|Q_2 \cap Q'| = 2k+1 = |Q_2 \cap Q_1|$) as well as $P_{3b}(Q_2, Q, B)$ (since $|Q_2 \cap Q \cap Q_1| = k+1$). Hence, $\boldsymbol{RQS} = \{Q, Q', Q_2, Q_1\}$ is a refined quorum system, where Q_1 (resp., Q_2) is a class 1 (resp., class 2) quorum.

Example 1. Figure 3 depicts a simple illustration of a refined quorum system for the *1-bounded threshold adversary* $\boldsymbol{B^1}$: 4 quorums are involved. As depicted by the example, the cardinality of a quorum is not always a good indication of its class: it is the intersection with others that matters. Quorum Q_1 contains 5 elements and is a class 1 quorum, while quorum Q' contains 6 elements yet is only a class 3 quorum.

In the following, we consider that an adversary $\boldsymbol{B}$ for a set of processes S contains all subsets of S that can simultaneously be Byzantine. In our description, a process that simply fails by crashing is not called Byzantine. We also denote by $\boldsymbol{Q_i}$ the set of subsets of S that contains all subsets of S that contain all but at most i elements of S, i.e., $\boldsymbol{Q_i} = \{P | P \subseteq S \wedge |P| \geq |S| - i\}$.

Example 2. Consider the case where: (a) $\boldsymbol{B} = \{\emptyset\}$, (b) $\boldsymbol{QC_1} = \boldsymbol{QC_2} = \emptyset$ and (c) $\boldsymbol{RQS} = \boldsymbol{Q_{\lfloor (|S|-1)/2 \rfloor}}$. In other words, every majority subset of S is a quorum. Property 1 is trivially satisfied. So are Properties 2 and 3, since $\boldsymbol{QC_1} = \boldsymbol{QC_2} = \emptyset$. This quorum system is typically used when devising algorithms that tolerate (a minority of) crash-failures, e.g., [4, 8, 16, 34, 44, 50].

Example 3. Consider the case of an adversary $\boldsymbol{B}^{\lfloor(|\boldsymbol{S}|-1)/3\rfloor}$, where (a) $\boldsymbol{QC_1}=\boldsymbol{QC_2}=\emptyset$ and (b) $\boldsymbol{RQS}=\boldsymbol{Q}_{\lfloor(|\boldsymbol{S}|-1)/3\rfloor}$. In this case, each quorum contains more than two thirds of processes and Property 1 is satisfied. Properties 2 and 3 are also satisfied (since $\boldsymbol{QC_1}=\boldsymbol{QC_2}=\emptyset$). Such a quorum system is typically used to tolerate (up to one third of) Byzantine failures, e.g., [7, 10, 42, 45].

Example 4. A refined quorum system for which $\boldsymbol{QC_1}=\boldsymbol{QC_2}=\emptyset$ is a *dissemination quorum system* in the sense of [40]. In [40], dissemination quorum systems were used to build SWMR regular storages [33] of authenticated (also called self-verifying) data. On the other hand, a refined quorum system in which $\boldsymbol{QC_1}=\emptyset$ and $\boldsymbol{QC_2}=\boldsymbol{RQS}$ is a *masking quorum system* in the sense of [40]. These systems have been used to build SWMR safe storages [33] of unauthenticated data. Both safe and regular semantics are weaker than atomic [33] which we target with RQS.

So far, in examples 2-4, we considered refined quorum systems in which $\boldsymbol{QC_1}=\emptyset$. In the rest of the paper, we study the more general case where $\boldsymbol{QC_1}\neq\emptyset$. This is the case where our refined quorum systems capture both the resilience and the best-case complexity dimensions of distributed algorithms.

Example 5. Consider the case of a refined quorum system where $\emptyset\neq\boldsymbol{QC_1}=\boldsymbol{QC_2}$. Such a RQS corresponds to the quorum system used in [36] for the specific case $\boldsymbol{B}=\{\emptyset\}$, to devise a consensus algorithm that tolerates asynchronous periods and a threshold t of process (crash) failures, yet expedites decisions in best-case scenarios. In fact, although not used in the algorithm, the idea of a *fast* quorum (class 1 quorum in our terminology) was used to explain its logic. In the special case of an adversary $\boldsymbol{B^k}$, where (a) $\boldsymbol{RQS}=\boldsymbol{Q_t}$, and (b) $\boldsymbol{QC_1}=\boldsymbol{QC_2}=\boldsymbol{Q_q}$ ($q\leq t$), Property 2 is satisfied if $|S|>2q+t+2k$ and Property 1 is satisfied if $|S|>2t+k$. These inequalities correspond to Lamport's lower bounds for "asynchronous" consensus [35].

The special case of this RQS where $k=q=t$ (i.e., where $\boldsymbol{QC_1}=\boldsymbol{RQS}$) corresponds to the quorum system used in [1, 41]. In this special case, RQS is built around a set containing $|S|>5t$ servers and where every quorum is a class 1 quorum. Both [41] and [1] showed how to achieve optimal consensus latency in synchronous periods despite t server failures using $|S|=5t+1$ servers. From the RQS perspective, the latency optimal features of these algorithms are simple to explain — in best-case scenarios, these algorithms were always able to operate on class 1 quorums. In this paper, we consider a more general notion of RQS that does not impose any penalty on the total number of servers while allowing for optimally resilient implementations.

Example 6. May be even more interesting is the case where $\emptyset\neq\boldsymbol{QC_1}\neq\boldsymbol{QC_2}\subseteq\boldsymbol{RQS}$ (e.g., Fig. 3), especially when $\boldsymbol{RQS}$, $\boldsymbol{QC_1}$ and the adversary are defined as in Example 5, $\boldsymbol{QC_2}=\boldsymbol{Q_r}$, and $0\leq q<r\leq t$. In other words, each quorum contains at least $|S|-t$ processes, while class 1 (resp., class 2) quorums contain at least $|S|-q$ (resp., $|S|-r$) processes. $\boldsymbol{RQS}$ satisfies (i) Property 1 if $|S|>2t+k$, (ii) Property 2 if $|S|>t+2k+2q$, and (iii) Property 3 if $|S|>t+r+k+min(k,q)$, i.e., $\boldsymbol{RQS}$ is a refined quorum system if $|S|>t+k+max(t,k+2q,r+min(k,q))$. This RQS corresponds to the quorum system used in [12, 20], and later in [52], as we detail below.

An important instantiation of this quorum system is the one with $|S|=3t+1$ processes, out of which t may be Byzantine ($k=t$), and where all quorums are class 2 quorums ($r=t$), whereas the set that contains all servers is a class 1 quorum ($q=0$). This exemplifies a quorum system that allows combining latency optimality with optimal resilience ($|S|=3t+1$) in a Byzantine context: optimal best-case latency can be achieved when all servers are available (class 1 quorum), whereas graceful degradation (next best possible latency) is possible if only a class 2 quorum is available. This combination is at the heart of the consensus algorithm of [12]. On the other hand, [20] employed the mentioned quorum system to present the first optimally resilient atomic storage algorithm that allows single round-trip operations (when class 1 quorum is accessed). Interestingly, [52] introduced the distinction between r and t (i.e., allowing for $r < t$) and the additional step in graceful degradation of consensus latency which can be interpreted as the distinction between class 2 and class 3 quorums.

Example 7 (Intuition behind Property 3 of RQS). As we just discussed, our RQS notion was implicitly used, in partial forms, in various distributed objects implementations. However, all examples we described assumed threshold adversary. This does not explain all the subtleness of RQS, notably of its Property 3, that becomes evident only when the general adversary structure is assumed.

Although the intuition behind Property 3 of RQS is not obvious, it should not be surprising that Property 3 is important to allow implementations that achieve both the best possible latency (e.g., 1 round in storage) and the next best possible latency (2 rounds in case of storage). Indeed, consider the following example in which there are 6 servers $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, with adversary structure given by

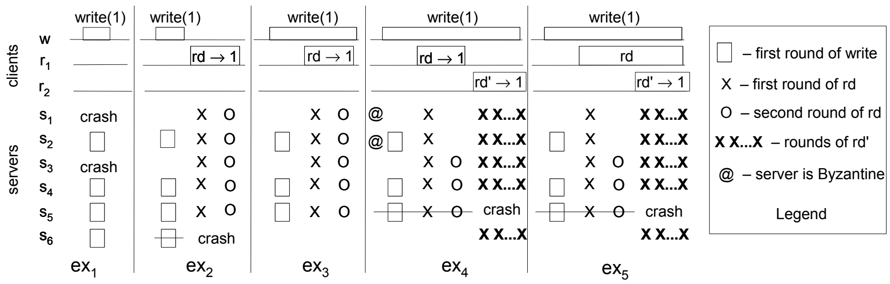
$$\boldsymbol{B} = \{\emptyset, \{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_1, s_2\}, \{s_3, s_4\}, \{s_2, s_4\}\}$$

with 3 quorums, $\boldsymbol{RQS} = \{Q_1, Q_2, Q_2'\}$, where $Q_1 = \{s_2, s_4, s_5, s_6\}$, $Q_2 = \{s_1, s_2, s_3, s_4, s_5\}$ and $Q_2' = \{s_1, s_2, s_3, s_4, s_6\}$. It is not very difficult to verify that Q_1 is a class 1 quorum, where Q_2 and Q_2' are class 2 quorums. Figure 4(a) depicts several executions of a possible best-case latency efficient atomic storage algorithm built over this RQS.

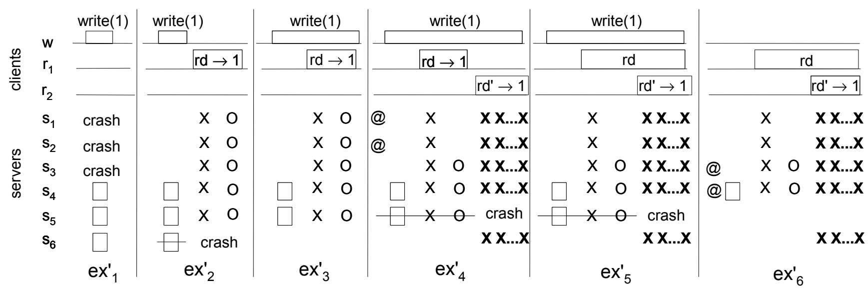
In execution ex_1 synchronous $\mathsf{write}(1)$ (we denote by wr) must complete in a single round since it accesses a class 1 quorum Q_1. In ex_2, wr completes as in ex_1 (although s_1 and s_3 are correct). Therefore, in ex_2, synchronous and uncontended read rd must return value 1 after 2 rounds of communication with servers from Q_2. Moreover, r_1 cannot distinguish ex_2 from ex_3 in which wr is slow, concurrent with rd and does not reach s_6. If we extend ex_3 such that s_5 crashes and servers from $B_{12} = \{s_1, s_2\}$ are Byzantine and "forget" about round 2 of rd, we obtain ex_4. In ex_4, rd' must return 1, but, at first, it is not clear that rd' should return the value in ex_4 after communicating only with servers from Q_2'. However, rd' in ex_4 is indistinguishable from rd' in ex_5 in which rd is simply slow and in which rd' must return a value (after some number of rounds) since all servers from Q_2' are correct (hence in both ex_4 and ex_5, rd' must return 1).

However, notice that, in ex_5, reader r_2 has (just) enough information to return 1, only because Property 3 of RQS holds. For example, since $B_{34} = Q_2 \cap Q_2' \setminus B_{12} = \{s_3, s_4\} \in \boldsymbol{B}$, $P_{3a}(Q_2, Q_2', B_{12})$ does not hold and consequently neither does $P_{3a}(Q_2, Q_2', B_{34})$. Hence $P_{3b}(Q_2, Q_2', B_{34})$ must hold, i.e., $Q_1 \cap Q_2 \cap Q_2' \nsubseteq B_{34}$. In our case $Q_1 \cap Q_2 \cap Q_2' \setminus B_{34} = \{s_2\}$, i.e., server s_2 (that was accessed in the first round of wr) is crucial for the ability of reader r_2 to return 1 in rd'.

Indeed, consider the case of a "broken" RQS: $\boldsymbol{RQS_b} = \{Q_{1b}, Q_2, Q_2'\}$, where $Q_{1b} = Q_1 \setminus \{s_2\} = \{s_4, s_5, s_6\}$. $\boldsymbol{RQS_b}$ is "broken" in a sense that it violates Property 3 (but not the Properties 1 and 2): a) $P_{3a}(Q_2, Q_2', B_{34})$ does not hold (like in $\boldsymbol{RQS}$) and b) since $Q_{1b} \cap Q_2 \cap Q_2' \subseteq B_{34}$, $P_{3b}(Q_2, Q_2', B_{34})$ does not hold either (unlike in $\boldsymbol{RQS}$). Executions ex_1' to ex_5' depicted in Figure 4(b) assume $\boldsymbol{RQS_b}$ and are respectively obtained from executions ex_1 to ex_5 when Q_1 is replaced with Q_{1b} (i.e., when wr does not write in s_2). Then, we can construct execution ex_6' similar to ex_5' (see Fig. 4(b)) in



(a) Executions that assume a genuine RQS



(b) Executions that assume a "broken" RQS that violates Property 3

Fig. 4. Intuition behind Property 3 of RQS; executions of a possible atomic storage implementation.

which (i) servers from B_{34} are Byzantine, (ii) s_5 is slow and (iii) there is no wr (i.e., value 1 is never written). However, reader r_2 cannot distinguish ex'_6 from ex'_5 and returns 1 in ex'_6, although 1 was never written. This violation of atomicity is a direct consequence of the violation of Property 3.

Finally, notice that the server s_2 in $Q_1 \cap Q_2 \cap Q'_2 \setminus B_{34}$ (in the genuine $\boldsymbol{RQS}$) would not be important if $P_{3a}(Q_2, Q'_2, B_{12})$ held, i.e., if B_{34} was not in $\boldsymbol{B}$. Then, in any execution, there would be at least one benign server in B_{34} and the reader would not have to worry about intersections with class 1 quorums.

In the following, we present two new algorithms that make full and explicit use of our notion of RQS.

3 Atomic storage

We show in this section how to use a refined quorum system to wait-free [24] implement the abstraction of a single-writer multi-reader (SWMR) atomic [33] storage with optimal resilience and complexity. Optimal resilience means here tolerating the maximal number of server failures while still ensuring wait-freedom in the face of contention and asynchrony (worst-case conditions). On the other hand, optimal complexity in our context means minimal operation latency in periods of synchrony and contention-freedom (best-case conditions). Our storage algorithm tolerates Byzantine servers yet does not rely on any data authentication primitive.[2]

In the following, and after few preliminaries on the model underlying our storage algorithm, we overview our algorithm and then state its optimality. This includes establishing a new tight bound on efficient atomic storage implementations. The full correctness proof of our atomic storage implementation is given in Appendix A.

3.1 Model

Processes and channels. We model processes as a deterministic I/O automata [39]. Processes are interconnected with point-to-point communication channels. For ease of presentation, we assume a global clock that is not accessible to processes. The state of the communication channel between processes p and q is modeled as a set $mset_{p,q} = mset_{q,p}$ containing messages that are sent but not yet received. We assume that every message m has two tags which identify the sender and the receiver of the message.

We model a distributed algorithm as a collection of automata A_p, each assigned to a process p. A computation of a process p proceeds in *steps* of A_p. A *correct* process p is one that executes an infinite number of steps of A_p. A process fails by *crashing* if it executes a finite number of steps. We say that a process is *benign* if it is correct or fails by crashing. A step of A is denoted by a pair of process id and message set $< p, M >$ (M might be $\emptyset$). In step $sp =< p, M >$, a benign process p atomically does the following (we say that p *takes* step sp): (1) (*receive substep*) p removes the messages of M from $mset_{p,*}$ (we also say: p *receives* the messages of M), (2) (*computation substep*) p applies M and its current state st_p to A_p, which outputs a new state st'_p and a set of messages M' to be sent, and p adopts st'_p as its new state and (3) (*send substep*) puts the output messages (M') in $mset_{p,*}$ (we also say: p *sends* the messages of M'). We assume that local computation takes negligible time, i.e., the time necessary for a benign process to take a step is negligible.

A *Byzantine* process p_B can perform arbitrary *actions*: (1) p_B can remove/put an arbitrary message m from/into $mset_{p_B,*}$ at an arbitrary time t,[3] and (2) p_B can change its state in an arbitrary manner. We say that a process is *faulty* if it is Byzantine or if it fails by crashing.

Given any algorithm A, an *execution* of A is an infinite sequence of steps of A taken by benign processes, and actions of Byzantine processes, such that the following properties hold for every benign process p: (1) initially, for each benign process q, $mset_{p,q} = \emptyset$, (2) the current state in the first step of p is a special state *Init*, and (3) for each step $< p, M >$ of A, and for every message $m \in M$, p is the receiver of m and $\exists q, mset_{p,q}$ that contains m immediately before the step $< p, M >$ is taken. A *partial execution* is a finite prefix of some execution. A (partial) execution ex *extends* some partial execution ex' if ex' is a prefix of ex. At the end of a partial execution, all messages that are sent but not yet received are said to be *in transit.*

We assume that the system is asynchronous: there is no bound on communication delays. The system *may be* synchronous during certain periods of time. We say that the *system is synchronous* during time interval $[t, t']$ if there is a constant Δ ($\Delta > 0$) known to all correct processes, such

[2] Although powerful, data authentication primitives do not provide deterministic guarantees, and might require (1) an infrastructure for key management (for solutions based on symmetric cryptography, e.g., [6]), or (2) non-negligible complexity of data encryption (e.g., [47]). These typically introduce overhead that one would like to avoid, especially in the best-case scenarios.

[3] Informally, we assume that no (benign) process uses data authentication.

that, for every two correct processes p_1 and p_2, every message sent by p_1 at time t_1 such that $[t_1, t_1 + \Delta] \subset [t, t']$, is received by p_2 at latest in the first receive substep taken by p_2 after $t_1 + \Delta$.

Finally, we assume that communication channels are *reliable*, i.e., if there is a step sp_p in which some correct process p sends a message m to another correct process q, then (eventually) q takes a step sp_q in which q receives m.

Distributed storage. A distributed storage (or, simply, storage) can be viewed as a read/write abstraction implemented by a finite set of processes called *servers*, and a distinct, potentially unbounded, set of processes called *clients*. We assume that the set of clients has two distinct subsets, a singleton *writer* and a set of *readers*. We assume clients and servers are connected with point-to-point channels (defined as in Section 3.1). No channels are assumed among servers.

An atomic storage provides the illusion of sequential accesses by ensuring the atomicity of read/write operations [25,33] (when there is no risk of ambiguity we say *operation* when we should be saying *operation execution*). We focus on *wait-free* [24] atomic storage algorithms in which every read/write operation invoked by a correct client eventually completes.

Clients access the storage through two operations: (1) write(v) (invoked by the writer), to write a value v in the storage, and (2) read() (invoked by readers), to read the value from the storage. We do not explicitly model the invocation and response steps of the storage operations. We simply say that an operation op invoked by client c is *complete* if c takes a response step for op. We assume that all benign servers are initialized with an initial value of the storage $\perp$, that is not in domain $\mathbb{D}$ of valid inputs of a write operation (we denote the set $\mathbb{D} \cup \{\perp\}$ by $\mathbb{D}_\perp$). No client c invokes a new operation before all operations previously invoked by c have completed.

Denote the time when operation op is invoked by $t_{op_{inv}}$. Denote the time when op completes by $t_{op_{resp}} > t_{op_{inv}}$. We say that a complete operation op' *precedes* operation op if $t_{op'_{resp}} < t_{op_{inv}}$ (we also say *op follows* op'). For two operations op and op', if neither op precedes op', nor op' precedes op, we say op and op' are *concurrent*. We say that an operation op is *uncontended* if op is not concurrent with any write operation. Moreover, we say that op is *synchronous* if the system is synchronous during the period between the invocation and completion of op.

Denoting the set of servers by S, and the adversary by $\boldsymbol{B}$, we construct a refined quorum system $\boldsymbol{RQS}$ (obeying the properties defined in Section 2) known to all clients. In the above, we assume that, for any execution ex, $B_{ex} \in \boldsymbol{B}$, where B_{ex} contains all servers Byzantine in ex. Moreover, any number of clients and servers may fail by crashing, as long as there is at least one quorum in $\boldsymbol{RQS}$ that contains only correct servers.

Round-based storage algorithms. Our algorithm is *round-based*, i.e., each operation op (read or write) proceeds in series of *communication round-trips* (or, simply, *rounds*). In each round of op,

1. Client c sends messages to a subset of servers (possibly all servers).
2. Servers on receiving such a message reply to client c before receiving any other messages. More precisely, any server s_i on receiving a message m in step $sp1 = \langle s_i, M \rangle$ ($m \in M$), where m is sent by the client c, replies to m either in step $sp1$ itself, or in a subsequent step $sp2$, such that s_i does not receive any message in any step between $sp1$ and $sp2$ (including $sp2$). Intuitively, this requirement forbids the server to wait for some other message before replying to m.
3. upon receiving a sufficient number $k \geq 0$ of such replies, a client completes a round. The decision on whether and when a client should proceed to the next round is algorithm-specific.

In short, in round-based algorithms, servers can send messages to clients only in response to some particular messages received from a client. Since we assume that servers (resp., clients) are not interconnected with communication channels, no other messages are exchanged in a round based algorithm.

We express the latency (complexity) of an operation *op* of a round-based algorithm in terms of the number of rounds between the invocation and the completion of *op*. Let $\boldsymbol{P}$ be any set of subsets of S.

Definition 3. $(m, \boldsymbol{P})$**–fast *storage algorithm*.** *Consider any synchronous and uncontended operation op invoked by a correct client. We say that a storage algorithm A is* $(m, \boldsymbol{P})$–fast *if in every execution of A in which some set* $P \in \boldsymbol{P}$ *contains only correct servers, op completes in at most m rounds, without using data authentication.*

3.2 Atomic storage algorithm

Overview. Our storage algorithm is $(m, \boldsymbol{QC_m})$*–fast* for all $m \in \{1, 2, 3\}$. Note that this implies that, in our algorithm, all synchronous and uncontended operations complete in at most 3 rounds. The pseudocode of the algorithm is given in Figures 5, 6 and 7.

The writer pseudocode (Figure 5) is simple, thanks to the underlying RQS. A write consists of at most three rounds of interactions between the writer and servers. If write is synchronous and a class i quorum is correct, write completes in i rounds ($i \in \{1, 2, 3\}$). The subtle part of the writer code is that the writer keeps track of class 2 quorums that respond in the first round, in order to ensure that the writer communicated with the same class 2 quorum in both rounds in the case of a 2-round write.

Server pseudocode is given in Figure 6. In order to simplify our algorithm, we assume that servers store the entire history of the shared variable they are implementing; we discuss this further in Section 5.

Finally, reader pseudocode is given in Figure 7. While it is more involved than those of writer/servers, it follows the structure of many other atomic storage algorithms (e.g., [4]). Namely, it consists of two parts: (1) the part that implements *regular* [33] storage (lines 20-35, along with the predicates defined in lines 3-9) and (2) the part that prevents read inversion and enforces atomicity, in which the readers write value back to a "sufficient" number of servers (lines 40-49, with predicates in lines 1-2). The key feature of the first part of the read algorithm is that it completes in a single round, if the read is synchronous and uncontended. The second part of algorithm then evaluates RQS intersections and the responses from servers received in the first round of such a read, to: a) skip the writeback if a class 1 quorum was accessed, b) perform 1 (resp., 2) rounds of writeback in case a class 2 (resp., 3) quorum was accessed.

In the remainder of this section, we first explain the details behind the write operation and then the details of read. The correctness proof of our atomic storage algorithm is postponed to Appendix A. The critical part of the proof consists of several short lemmas. The main theorems (that make use of the above mentioned lemmas), despite being long, are easy to follow, since these are mainly case-by-case analysis, where the intersection properties of RQS allow the critical lemmas to be easily applied.

Write operation. A write consists of at most three rounds. The writer maintains a monotonically increasing local timestamp ts that is assigned to the written value v and sent to servers in each

Initialization:
0: $ts := 0$; $timeout := 2\Delta$; $\boldsymbol{QC'_2} := \emptyset$

write(v) **is** {
1: **inc**(ts)
2: round(1)
3: **if** acks received from some class 1 quorum **then return**(OK)
4: **forall** $Q_2 \in \boldsymbol{QC_2}$
5: **if** acks received from Q_2 **then** $\boldsymbol{QC'_2} := \boldsymbol{QC'_2} \cup \{Q_2\}$
6: round(2)
7: **if** acks received from some quorum in $\boldsymbol{QC'_2}$ **then return**(OK)
8: $\boldsymbol{QC'_2} := \emptyset$; round(3)
9: **return**(OK) }

round(i) **is** {
10: send $\mathsf{wr}\langle ts, v, \boldsymbol{QC'_2}, i\rangle$ to all servers
11: **if** $i < 3$ **then trigger**($timeout$)
12: wait for (reception of $\mathsf{wr\_ack}\langle ts, i\rangle$ from some quorum) **and** (expiration of $timeout$) }

Fig. 5. Atomic storage algorithm: writer code

round (for simplicity, we sometimes say that the writer writes a pair $\langle ts, v\rangle$). More precisely, in every round rnd, the writer sends a $\mathsf{wr}\langle ts, v, \boldsymbol{QC'_2}, rnd\rangle$ message containing value v to be written, along with a timestamp ts and a set of quorums (i.e., quorum ids) $\boldsymbol{QC'_2}$ to all servers (this set is empty in rounds 1 and 3 and only used in round 2, as explained below). In every round, the writer awaits acks from some quorum and, in the first two rounds, the expiration of the timer set to 2Δ.

If the writer receives acks from some class 1 quorum by the expiration of the timer, the write terminates. Otherwise, the writer proceeds to round 2. If, in round 1, the writer received acks from some class 2 quorums, the ids of these quorums are added to $\boldsymbol{QC'_2}$ (lines 4-5, Fig. 5). If the writer receives again acks from some quorum from $\boldsymbol{QC'_2}$ in round 2 (line 7, Fig. 5), the write terminates at the end of round 2. Finally, if this is not the case, the writer proceeds to round 3 and completes at the end of this round, upon reception of round 3 acks from any quorum.

Upon receipt of the message $\mathsf{wr}\langle ts, v, \boldsymbol{QC'_2}, rnd\rangle$, a server s_i stores the received data in its $history_i$ matrix, by storing $history_i[ts, j].pair = \langle ts, v\rangle$ for all $j, 1 \le j \le rnd$ and by adding $\boldsymbol{QC'_2}$ to $history_i[ts, rnd].sets$ (see Fig. 6). Then, a server sends an ack to the client.

Initialization:
$history_i[*, *].pair := \langle 0, \perp\rangle$; $history_i[*, *].sets := \emptyset$

upon reception of a $\mathsf{wr}\langle ts, v, \boldsymbol{QC'_2}, rnd\rangle$ message from some client c **do**
for $m = 1$ to rnd **do**
if $history_i[ts, m] = \langle\langle 0, \perp\rangle, \emptyset\rangle$ **or** $history_i[ts, m] = \langle\langle ts, v\rangle, *\rangle$ **then**
$history_i[ts, m].pair := \langle ts, v\rangle$
if $m = rnd$ **then** $history_i[ts, m].sets := history_i[ts, m].sets \cup \boldsymbol{QC'_2}$
send $\mathsf{wr\_ack}\langle ts, rnd\rangle$ to c

upon reception of a $\mathsf{rd}\langle tsr, rnd\rangle$ message from the reader r_j
send $\mathsf{rd\_ack}\langle tsr, rnd, history_i\rangle$ to the reader r_j

Fig. 6. Atomic storage algorithm: server s_i code

Definitions:

1: $BCD(c,1,R) ::= \exists Q_1 \in \boldsymbol{QC_1},\ \exists Q_R \in \boldsymbol{QC_R},\ \exists \boldsymbol{Set} \subseteq \boldsymbol{QC_2} \cup \{\emptyset\}$:
$(Q_1 \cap Q_R \subseteq \{s_i \in S \mid history[i, c.ts, R] = \langle c, \boldsymbol{Set}\rangle\}) \wedge ((R \neq 2) \vee (Q_R \in \boldsymbol{Set}))$

2: $\boldsymbol{BCD}(c,2,R) ::= \{Q_2 \in \boldsymbol{QC_2'} \mid \exists Q_R \in \boldsymbol{QC_R}\colon Q_R \cap Q_2 \subseteq \{s_i \in S \mid history[i, c.ts, R].pair = c\}\}$

3: $valid_1(c,Q) ::= \exists T \subseteq Q, \forall s_i \in T : (T \notin \boldsymbol{B}) \wedge (history[i, c.ts, 1].pair = c)$
4: $valid_2(c,Q) ::= \exists s_i \in Q : history[i, c.ts, 2].pair = c$
5: $valid_3(c,Q) ::= \exists Q_2 \in \boldsymbol{QC_2}, \exists B \in \boldsymbol{B}, \forall s_i \in Q_2 \cap Q \setminus B,\ \exists\, \boldsymbol{Set_i} \subseteq \boldsymbol{QC_2}$:
$(P_{3b}(Q_2, Q, B)) \wedge (history[i, c.ts, 1] = \langle c, \boldsymbol{Set_i}\rangle) \wedge (Q_2 \in \boldsymbol{Set_i})$

6: $invalid(c) ::= \exists Q \in \boldsymbol{Responded}\colon \neg(valid_1(c,Q) \vee valid_2(c,Q) \vee valid_3(c,Q)) \vee (c.ts > highest\_ts)$

7: $read(c,i) ::= \exists rnd \in \{1,2\} : history[i, c.ts, rnd].pair = c$
8: $safe(c) ::= \{s_i \in S \mid read(c,i)\} \notin \boldsymbol{B}$
9: $highCand(c) ::= \forall c' \in \mathbb{N}_0 \times \mathbb{D}_\perp, \forall s_i \in S : read(c', i) \wedge (c'.ts > c.ts) \Rightarrow invalid(c')$

Initialization:

10: $timeout := 2\Delta$; $history[*,*,*] := \langle\langle 0, \perp\rangle, \emptyset\rangle$; $highest\_ts := 0$; $read\_no := 0$

read() is {
20: $read\_rnd := 0$; $\boldsymbol{QC_2'} := \emptyset$; $\boldsymbol{Responded} := \emptyset$
21: **inc**($read\_no$)
22: **repeat**
23: **inc**($read\_rnd$)
24: **if** $read\_rnd = 1$ **then trigger**($timeout$)
25: send rd$\langle read_no, read_rnd\rangle$ to all servers
26: **wait for** receive rd_ack$\langle read_no, read_rnd, *\rangle$ from some quorum
27: **if** $read\_rnd = 1$ **then**
28: **wait for** expiration of $timeout$
29: $highest\_ts :=$ highest timestamp $hts \in \mathbb{N}_0$ such that $\exists c \in \mathbb{N}_0 \times \mathbb{D}_\perp, \exists s_i \in S : read(c,i) \wedge c.ts = hts$
30: **forall** $Q_2 \in \boldsymbol{QC_2}$
31: **if** acks received from Q_2 **then** $\boldsymbol{QC_2'} := \boldsymbol{QC_2'} \cup \{Q_2\}$
32: **endif**
33: $C := \{c \in \mathbb{N}_0 \times \mathbb{D}_\perp \mid (safe(c) \wedge highCand(c))\}$
34: **until** $C \neq \emptyset$
35: $c_{sel} := c \in C : (\forall c' \in C : c.ts \geq c'.ts)$

40: **if** $(\exists i \in \{1,2,3\} : BCD(c_{sel}, 1, i))$ **and** $(read\_rnd = 1)$ **then return**($c_{sel}.val$)

41: **if** $(\exists i \in \{1,2,3\} : \boldsymbol{BCD}(c_{sel}, 2, i) \neq \emptyset)$ **and** $(read\_rnd = 1)$) **then**
42: **if** $(\exists i \in \{2,3\} : \boldsymbol{BCD}(c_{sel}, 2, i) \neq \emptyset)$ **then writeback**$(2, c_{sel}, \emptyset)$
43: **trigger**($timeout$)
44: **writeback**$(1, c_{sel}, \boldsymbol{BCD}(c_{sel}, 2, 1))$
45: **wait for** expiration of $timeout$
46: **if** acks received from some quorum from $\boldsymbol{BCD}(c_{sel}, 2, 1)$ **then return**($c_{sel}.val$)
47: **writeback**$(2, c_{sel}, \emptyset)$
48: **endif**

49: **writeback**$(1, c_{sel}, \emptyset)$; **writeback**$(2, c_{sel}, \emptyset)$

50: **upon** reception of rd_ack$\langle read_no, read_rnd, history_i\rangle$ from server s_i **do**
51: $history[i] := history_i$

52: **upon** received at least one rd_ack message from every server s_i in some quorum $Q \in \boldsymbol{RQS}$ **do**
53: $\boldsymbol{Responded} := \boldsymbol{Responded} \cup Q$

writeback($round, c, \boldsymbol{Set}$) is {
60: send wr$\langle c.ts, c.val, \boldsymbol{Set}, round\rangle$ message to all servers
61: **wait for** reception of wr_ack$\langle c.ts, round\rangle$ message from some quorum
62: **if** $round = 2$ **then return**($c_{sel}.val$) }

Fig. 7. Atomic storage algorithm: reader code

Read operation. As we already mentioned, reader code of our algorithm (given in Figure 7 to which we refer in the following, unless stated otherwise) consists of two parts: (1) the part that implements *regular* [33] storage (lines 20-35, with predicates in lines 3-9) and (2) the writeback part (lines 40-49, with predicates in lines 1-2).

In the first part of the read algorithm the reader selects the timestamp/value pair (in line 35), that contains the value that the reader is going to return after a possible writeback. The first part of the algorithm consists in one or more rounds in which the reader sends rd$\langle read_no, read_rnd\rangle$ (line 25), containing the unique id of a read $read\_no$ (to distinguish messages sent by the same reader in different operations) and the round number $read\_rnd$. A server replies to a rd message by sending the entire history of the shared variable in a rd_ack message in response (lines 8-9, Fig. 6). A round ends when the reader receives responses from all servers from some quorum Q (line 26). Specifically, in round 1, the reader: (a) also waits for the timer set to 2Δ to expire (lines 24 and 28), and (b) stores the ids of all class 2 quorums that responded to it in the set $\boldsymbol{QC'_2}$ (lines 30-32). In general, we say quorum Q responds in a read if a reader receives at least one rd_ack from every server in Q (lines 52-53). Remembering class 2 quorums that responded in round 1 will later reveal crucial for allowing 2-round best-case reads and single round best-case writes in the same implementation.

In the heart of the first part of the algorithm are predicates $valid_j$ (for $j \in \{1, 2, 3\}$), defined in lines 3-5. These predicates ensure that if some complete write (resp., read) operation op wrote (resp., selected) a pair $c = \langle c.ts, c.val\rangle$, in any read rd that follows op, for every quorum Q that responded in rd there is some j such that $valid_j(c, Q)$ holds. Therefore, predicate $invalid(c)$ (line 6) cannot hold in rd and, the reader cannot select a pair c_{sel} such that $c_{sel}.ts < c.ts$ in line 35 of rd; in other words, rd cannot return an older value than the one written/returned by op. The pair c_{sel} is selected in line 35, as the pair with the highest timestamp among all pairs c for which predicates $highCand(c)$ (line 9) and $safe(c)$ holds (line 8). Predicate $highCand(c)$ implies that all pairs with a higher timestamp are invalid, i.e., that there are no possibly newer values that ought to be considered. On the other hand, predicate $safe(c)$ guarantees that a selected value is not fabricated by Byzantine processes; roughly speaking, all servers from some set $T \notin \boldsymbol{B}$ need to confirm c before a reader can select it. This prevents fabrication since, in every execution, T contains at least one benign server.

On the other hand, the second part of the algorithm that ensures atomicity, is orchestrated around the outcome of a *Best-Case Detector* (BCD), defined by predicates in lines 1-2, and accessed RQS quorums. Roughly, BCD detects if a read operation is synchronous and uncontended. In the following, we explain the intuition behind the techniques used in our algorithm on the example of a one such uncontended and synchronous read rd.

Let wr be the last write that precedes rd and assume that wr wrote pair $c = \langle ts, v\rangle$ in R rounds, $R \in \{1, 2, 3\}$. Note that, by atomicity, rd must return v.

First, it is crucial to see that, in a synchronous and uncontended read like rd, the first part of the algorithm takes *only* a single round. Since rd is uncontended, during rd benign servers store values with timestamps only as high as ts. As we already intuited above, in rd, for every quorum Q that responds in round 1, there is some j such that $valid_j(c, Q)$ holds. To see this, notice that wr completed either in: (a) single round ($R = 1$) by accessing class 1 quorum Q_1, or (b) in more than one round ($R \in \{2, 3\}$). Then, it is not difficult to see that for every quorum Q, in case: (a) $valid_1(c, Q)$ holds (by Property 2 of RQS), whereas in (b) $valid_2(c, Q)$ holds (by Property 1 of RQS). Hence, $invalid(c)$ cannot hold at the end of round 1. Moreover, since rd is synchronous, it gets a response from at least one quorum Q_c containing only correct servers; hence, $safe(c)$

also holds. Considering possible pairs c' with higher timestamp than ts (which could have been reported by Byzantine servers only), it is not difficult to see that, for such c', none of the predicates $valid_j(c', Q_c)$ for $j \in \{1, 2, 3\}$ can hold. Hence, in the case of synchronous and uncontended read rd the reader selects a pair $c_{sel} = c$ written by the last preceding write in line 35 (set C in line 33 is a singleton in such a read), and the first part of the algorithm (lines 20-35) takes only a single round.

Then, the reader proceeds to the second part of the read algorithm (that guarantees atomicity, lines 40-49); basically, this is a sophisticated *writeback* procedure, based on the outcome of a BCD. The reader queries BCD with $c_{sel} = \langle ts, v\rangle$ as a parameter (line 40) and the outcome governs the remainder of the writeback procedure. Namely:

1. If the reader received acks from a class 1 quorum (containing only correct servers) in round 1, $BCD(c_{sel}, 1, R)$ holds (line 1) and rd completes at the end of round 1, *without* any writeback whatsoever (line 40). Recall here that R denotes the number of rounds in which wr completed and hence suggests the class of the quorum that was available to the writer. Notice that, by line 1, $BCD(c_{sel}, 1, R)$ holds only if there is a class 1 quorum Q_1 and a class R quorum Q_R such that *all* servers from $Q_1 \cap Q_R$ had received a round R wr message containing ts and v (either from the writer or some reader writing-back the pair c_{sel}) and responded to the reader. Since read rd is synchronous and uncontended, $BCD(c_{sel}, 1, R)$ is guaranteed to hold in case rd accesses a class 1 quorum of correct processes in the first round.
2. Else, if the reader received acks from some class 2 quorum(s) Q_2 (containing only correct servers) in round 1, then set $\boldsymbol{X}$=$\boldsymbol{BCD}(c_{sel}, 2, R)$ (line 2) is non-empty set of quorums (since $Q_2 \in \boldsymbol{X}$). Indeed, notice that $\boldsymbol{X}$=$\boldsymbol{BCD}(c_{sel}, 2, R)$ contains a set of all class 2 quorums Q_2 such that: (a) the reader received replies from Q_2 in round 1 of rd (i.e., $Q_2 \in \boldsymbol{QC'_2}$), and (b) there is a class R quorum Q_R such that *all* servers from $Q_2 \cap Q_R$ received the round R wr message containing ts and v. Since read rd is both synchronous and uncontended, $\boldsymbol{X}$ is guaranteed to contain all class 2 quorums (containing only correct processes) that replied to the reader in round 1.

   Since $\boldsymbol{X}$ is non-empty, rd proceeds to round 2 (or, in other words, the first round of the write-back procedure, line 41). If $R \in \{2, 3\}$, then the reader sends wr$\langle ts, v, \emptyset, rnd\rangle$, with $rnd = 2$ to all servers, waits for acks from some quorum and returns (lines 42 and 60-62). In this case, reader writesback with $rnd = 2$ since it knows that the writer already completed wrote the value to some quorum (writing the value to servers using wr$\langle ts, v, *, rnd\rangle$ message with $rnd = 2$ conveys that the client knows that all servers from some quorum have already stored pair $\langle ts, v\rangle$).

   Else, if $R = 1$, the first round of the writeback procedure (round 2 of rd) is more sophisticated, since the reader cannot be sure that all servers from some quorum already stored $\langle ts, v\rangle$ (recall here the executions depicted in Fig. 4, in Example 7, of Section 2.2). Namely, in this case, the reader: (a) triggers a timer (line 43), (b) sends wr$\langle ts, v, \boldsymbol{X}, 1\rangle$ to all servers (line 44), and (c) waits for acks until some quorum responds and the timer expires (lines 45 and 60-62). The uncontended and synchronous read completes at the end of round 2 (the first round of the writeback) only if the reader receives acks from some quorum from $\boldsymbol{X}$=$\boldsymbol{BCD}(c_{sel}, 2, 1)$ (line 46).

   Writing class 2 quorum ids contained in the $\boldsymbol{X}$, is crucial for allowing 2 round best-case reads to be combined with single round best-case writes. For example, if ex_5 of Figure 4 is applied to our algorithm, the reader in r_1 would be writing back the value in the second round of rd precisely as described above.

3. Otherwise, if no quorum from $\boldsymbol{X}$ replies, the second round of the writeback procedure (i.e., the third round of rd) is invoked (line 47). Note that the read takes at most 2 rounds in the second part of the algorithm (i.e., in lines 40-49). Hence, when the read is synchronous and uncontended, it completes in at most 3 rounds.

Finally, notice that while set C in line 33 is a singleton at the end of the first round of a synchronous and uncontended read (as already mentioned), this is not necessarily the case in a contended read. Namely, in such a read C might be empty, or even contain more than a single value (intuitively, the first part provides regular semantics, where multiple values can be returned). The following simple example illustrates this further.

Consider 4 servers ($S = \{s_1, s_2, s_3, s_4\}$), threshold adversary $\boldsymbol{B^1}$ (at most 1 server can be Byzantine) and an RQS formed of a class 1 quorum that contains all 4 servers and four class 2 quorums, each containing exactly 3 servers. Assume that server s_1 crashes at the beginning of an execution in which other servers in $Q = \{s_2, s_3, s_4\}$ are correct, and in which read rd is concurrent with two 2-round writes, wr_1 and wr_2, which write pairs $c_1 = \langle 1, v_1\rangle$ and $c_2 = \langle 2, v_1\rangle$, respectively. In the first round of rd, servers s_2 and s_3 respond with their initial histories (these servers "see" no write), while s_4 "sees" a complete 2-round write of pair c_1. It is not difficult to see that, at the end of round 1 of rd, $safe(\langle 0, \perp\rangle)$ and $valid_2(c_1, Q)$ hold, $C = \emptyset$ and $highest\_ts = 1$. Then, both wr_1 and wr_2 complete and, in the second round of rd, all servers from Q report that they "see" both rounds of both writes. Then, at the end of the second round of rd, $C = \{c_1, c_2\}$. Indeed, it is straightforward to see that both $safe(c_1)$ and $safe(c_2)$ hold. Moreover, $invalid(c_2)$ holds, because $c_2.ts = 2 > highest\_ts$, and hence $highCand(c_1)$ holds. However, $highCand(c_2)$ also trivially holds (since there is no pair c' and server s_i such that $read(c', i)$ holds and $c'.ts > c_2.ts$). Intuitively, in our algorithm, while $highest\_ts$ is used as a cutoff timestamp to help ensure wait-freedom, the reader still returns the latest written value whenever possible (in this case $c_2.val = v_2$).

3.3 Optimality

Consider the space of round-based storage algorithms. Let $\boldsymbol{Q}$, $\boldsymbol{Q^{(i)}}$ (for $i \in \{1, 2, 3\}$) be any sets of subsets of (the set of servers) S. We say that an algorithm A is $(\boldsymbol{Q}, \boldsymbol{B})$*–atomic*, if A wait-free implements an atomic SWMR storage despite the adversary $\boldsymbol{B}$ provided that in every execution of A, there is a set $Q \in \boldsymbol{Q}$ that contains only correct servers. The minimality of our RQS is captured via the following three theorems.

Theorem 1. *If an algorithm A is $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$–atomic, then $P1(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$ holds.*

Theorem 2. *If a $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$–atomic algorithm A is $(1, \boldsymbol{Q^{(1)}})$–fast, then $P2(\boldsymbol{Q^{(1)}}, \boldsymbol{Q^{(3)}}, \boldsymbol{B})$ holds.*

Theorem 3. *If a $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$–atomic algorithm A is both $(1, \boldsymbol{Q^{(1)}})$–fast (for some $\boldsymbol{Q^{(1)}} \neq \emptyset$) and $(2, \boldsymbol{Q^{(2)}})$–fast, then $P3(\boldsymbol{Q^{(1)}}, \boldsymbol{Q^{(2)}}, \boldsymbol{Q^{(3)}}, \boldsymbol{B})$ holds.*

As a corollary of Theorems 1–3, our atomic storage implementation of Figure 7 is optimally resilient and has optimal (best-case) complexity.

Theorem 1 has been established for the special case of threshold-based quorums and with an implicit notion of quorums in [42]. Moreover, a restricted form of Theorem 2 was proved in [20], which considered atomic storage implementations in which synchronous and uncontended read/write operations can complete in a single round, in the context of optimally resilient atomic storage implementations in the threshold-based hybrid failure model [49]. It is not very difficult to extend these bounds to the general adversary structure and the RQS setting. Theorem 3 is entirely novel, and particularly interesting, due to the unusual *or* condition that appears in Property 3 of RQS. In the following we prove Theorem 3.

Proof. Theorem 3 states that there is no $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$–*atomic* storage algorithm that is both $(1, \boldsymbol{Q^{(1)}})$–*fast* (for some $\boldsymbol{Q^{(1)}} \neq \emptyset$) and $(2, \boldsymbol{Q^{(1)}})$–*fast*, if Property 3 of RQS is violated. Assume by contradiction that such a storage algorithm A exists even if Property 3 of RQS is violated. Consider a simple SWMR storage algorithm with a single writer w and two distinct readers $w \neq r_1 \neq r_2 \neq w$. In the following, we denote by $\overline{X}$ the set $S \setminus X$, where X is any subset of the set S (recall that S denotes the set of all servers). Negating $P3(\boldsymbol{Q^{(1)}}, \boldsymbol{Q^{(2)}}, \boldsymbol{Q^{(3)}})$ (Property 3 of RQS) yields (having in mind $\boldsymbol{Q^{(1)}} \neq \emptyset$):

$$\exists Q_1 \in \boldsymbol{Q^{(1)}},\ \exists Q_2 \in \boldsymbol{Q^{(2)}},\ \exists Q \in \boldsymbol{Q^{(3)}},\ \exists B_1', B_2 \in \boldsymbol{B}\text{: } (Q_2 \cap Q \setminus B_1' = B_2) \wedge (Q_1 \cap Q_2 \cap Q \subseteq B_1').$$

In the following, we denote the set $Q_1 \cap Q_2 \cap Q$ by B_0 and $Q_2 \cap Q \cap B_1'$ by B_1. Having in mind that $\boldsymbol{B}$ is an adversary for S, it is straightforward to verify the following:

- $B_0, B_1 \subseteq B_1'$,
- $B_0, B_1 \in \boldsymbol{B}$, and
- $Q_2 \cap Q = B_1 \cup B_2$.

Moreover, since $B_0 \subseteq B_1'$ and $B_0 \subseteq Q_2 \cap Q$, we have $B_0 \subseteq B_1$. Hence, $Q_2 \cap Q \cap \overline{Q_1} = B_2 \cup (B_1 \setminus B_0)$.

To exhibit a contradiction, we construct several partial executions (sketched in Figure 8) of the algorithm A including one in which atomicity is violated. More specifically, in this particular partial execution, a read operation returns a value that was never written.

- Let ex_1 be the execution in which all servers from Q_2 are correct, while all others (i.e., those from $\overline{Q_2}$) fail by crashing at the beginning of the execution. Furthermore, let wr_1 be the write operation invoked at time t_1 by the correct writer w in ex_1 to write a value $v_1 \neq \perp$ in the storage. Moreover, assume that the system is synchronous in ex_1. Hence, wr_1 is synchronous and uncontended. Since A is $(2, \boldsymbol{Q^{(2)}})$–*fast*, wr_1 completes in ex_1, say at time t_1', in at most two communication rounds, after the writer receives the replies in round 2 from servers from Q_2.
- Let ex_1' be the partial execution that ends at t_1', such that ex_1' is identical to ex_1 up to time t_1', except that in ex_1' servers from $\overline{Q_2}$ do not crash, but, due to asynchrony, all messages sent by the writer to $\overline{Q_2}$ during wr_1 remain in transit. Since the writer cannot distinguish ex_1' from ex_1, wr_1 completes in ex_1', in two communication rounds, at time t_1'.
- Let the partial execution ex_2 extend ex_1' such that: (1) servers from $\overline{Q_1}$ crash at t_1', (2) rd_1 is a synchronous read operation invoked by the correct reader r_1 after t_1', and (3) no other operation is invoked (hence, rd_1 is uncontended). Since A is $(1, \boldsymbol{Q^{(1)}})$–*fast*, rd_1 completes in a single round (since a set Q_1 of servers is correct) at time t_2 and returns v_1. Moreover, let ex_2 end at t_2. All messages that were in transit in ex_1' remain in transit in ex_2.
- Let ex_2' be the partial execution identical to ex_2 except that in ex_2' servers from $\overline{Q_1}$ do not crash, but, due to asynchrony, the message sent from r_1 to servers in $\overline{Q_1}$ during rd_1 remains in transit in ex_2'. Since r_1 and all servers, except those from $\overline{Q_1}$, cannot distinguish ex_2' from ex_2, rd_1 completes in ex_2' in a single round, at time t_2, and returns v_1.

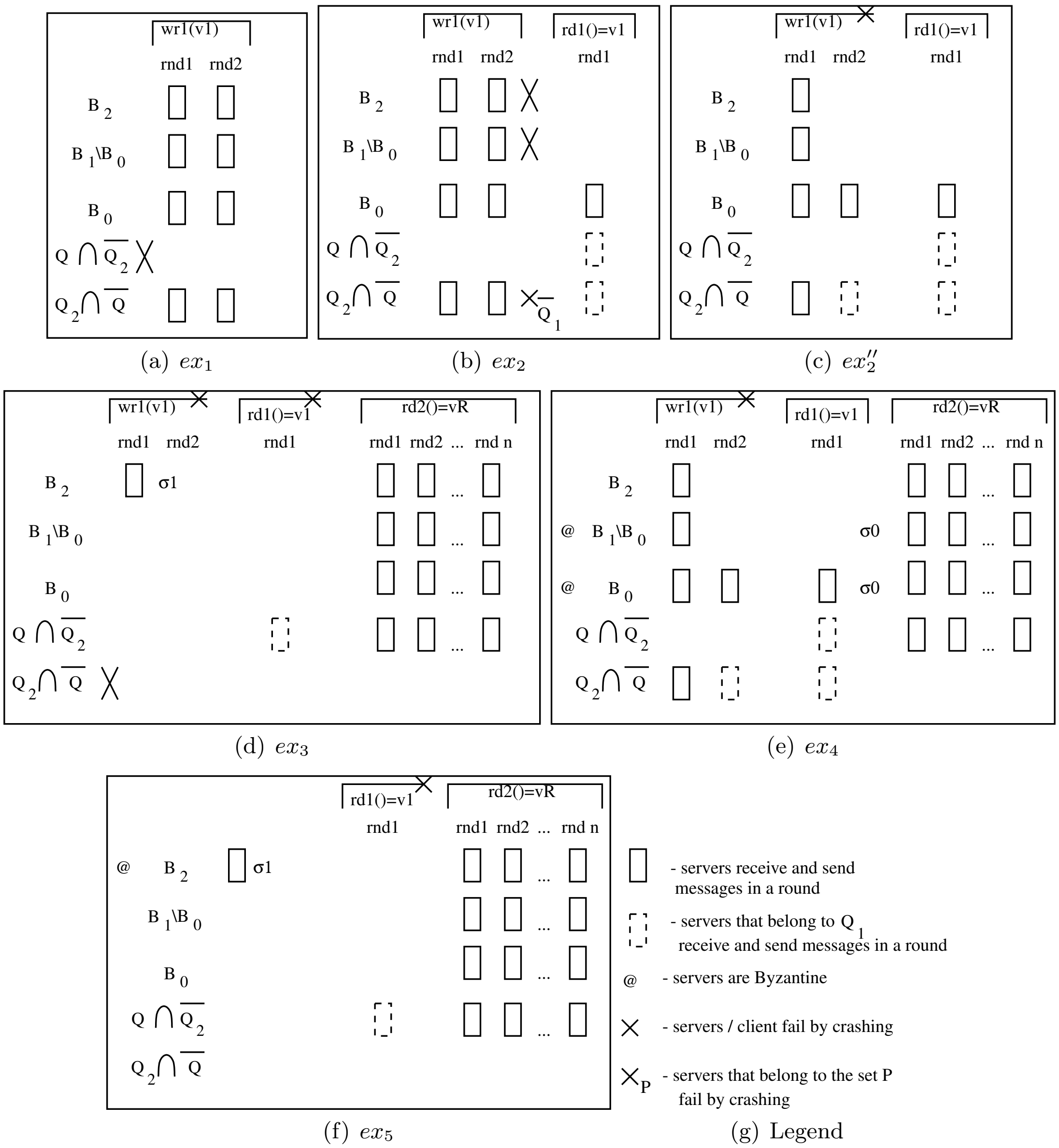


Fig. 8. Illustration of the partial executions used in the proof of Theorem 3. Only servers that belong to the set $Q_2 \cup Q$ are depicted.

– Let ex_2'' be the partial execution identical to ex_2' except that, in ex_2'': (1) the writer crashes during wr_1 and its round 2 messages are not received by any servers from $\overline{Q_1} \cup \overline{Q_2}$ (i.e., only servers from $Q_1 \cap Q_2$ receive the round 2 message from the writer). Note that all servers from the set $B_2 \cup (B_1 \setminus B_0)$ belong to $\overline{Q_1}$ and, hence, do not receive a round 2 message from the writer. Since r_1 and all servers, except those from $\overline{Q_1} \cap Q_2$, cannot distinguish ex_2'' from ex_2', rd_1 completes in ex_2'' at time t_2 and returns v_1.
– Consider now a partial execution ex_3 slightly different from ex_2'' in which the writer (resp., the reader r_1) crashes during the round 1 of wr_1 (resp., rd_1) such that the round 1 messages sent by the writer (resp., r_1) in wr_1 (resp., rd_1) are received only by servers from the set B_2 (resp., $Q \cap \overline{Q_2} \cap Q_1$). We refer to the state of servers that belong to the set B_2 *after* sending the reply to the round 1 message of wr_1 as to σ_1. In ex_3, all servers are correct except those from the set $\overline{Q}$ that fail by crashing at the beginning of partial execution ex_3. Assume that the writer crashes at time t_{fail_w} and that r_1 crashes at time $t_{fail_r} > t_{fail_w}$. Let rd_2 be a read operation invoked by the correct reader $r_2 \neq r_1$ at time $t_3' > max(t_{fail_r}, t_2)$. Since all servers from the set Q are correct in ex_3 and A is a $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$*–atomic* storage algorithm, rd_2 eventually completes, at some point in time t_3, after n communication rounds and returns value v_R.
– Let ex_4 be a partial execution identical to ex_2'' except that in ex_4: (1) a read operation rd_2 is invoked by the correct reader r_2 at t_3' (as in ex_3), (2) due to asynchrony all messages sent by servers from $\overline{Q}$ to r_2 are delayed until after t_3 (i.e., until after n^{th} round of rd_2) and (3) in ex_4, all servers from B_1 (and B_0, since $B_0 \subseteq B_1$) are Byzantine: these servers forge their state at time t_2 to σ_0 (the initial state of servers); otherwise, servers from B_1 obey the protocol (including with respect to the writer and the reader r_1). Note that r_2 and servers from $Q \setminus B_1 = B_2 \cup (Q \cap \overline{Q_2})$ cannot distinguish ex_4 from ex_3 and, hence, rd_2 completes in ex_4 at time t_3 (as in ex_3) and returns v_R. On the other hand, r_1 cannot distinguish ex_4 from ex_2''. Hence, rd_1 completes in a single round and returns v_1. By atomicity, since rd_1 precedes rd_2, v_R must equal v_1.
– Consider now the partial execution ex_5, identical to ex_3, except that in ex_5: (1) wr_1 is never invoked, (2) servers from B_2 are Byzantine in ex_5 and forge their state to σ_1 (see ex_3); otherwise, servers from B_2 send the same messages as in ex_3, and (3) servers from $\overline{Q}$ do not crash in ex_5, but, due to asynchrony, all messages sent from servers from $\overline{Q}$ to r_2 are delayed until after t_3 (i.e., n^{th} round of rd_2). The reader r_2 and servers from $Q \setminus B_2 = B_1 \cup (Q \cap \overline{Q_2})$ cannot distinguish ex_5 from ex_3, so rd_2 completes at time t_3 and returns v_R, i.e., v_1 (see ex_4). However, by atomicity, in ex_5, rd_2 must return $\perp$, the initial value of the atomic storage. Since $v_1 \neq \perp$, ex_5 violates atomicity.

Finally, notice that the assumption that Property 3 of RQS does not hold is critical in reaching a violation of atomicity using the above sequence of executions ex_1 to ex_5. Namely, if Property 3 holds, then $P_{3a}(Q_2, Q, B_1')$ holds (implying $B_2 = Q_2 \cap Q \setminus B_1' \notin \boldsymbol{B}$), in which case we cannot have ex_5, or $P_{3b}(Q_2, Q, B_1')$ holds (implying $B_0 \setminus B_1' \neq \emptyset$, i.e., $B_0 \nsubseteq B_1'$, where $B_0 = Q_1 \cap Q_2 \cap Q$), in which case we cannot have ex_4. □

4 Consensus

In this section, we give the second example of using RQS to obtain a novel implementation of an important abstraction, optimal in terms of resilience and best-case complexity. The example we consider here is that of implementing a *consensus* abstraction, which is at the heart of the state machine replication technique [32], widely considered for building general reliable services (beyond the storage abstraction).

The consensus algorithm we present tolerates Byzantine failures of processes and unbounded periods of asynchrony. In fact, it is the first consensus algorithm that tolerates an unbounded number of Byzantine proposers and learners (in practical state machine replication algorithms, unbounded number of proposers and learners would typically be translated into the unbounded number of clients). The algorithm is optimal in terms of resilience as well as complexity, matching the lower bounds of [35] and closing, we believe, a very important gap. The notion of complexity considered here is again best-case complexity for this is considered practically appealing on the one hand and, on the other hand, the worst-case complexity of a consensus algorithm that tolerates arbitrarily long periods of asynchrony is anyway unbounded. Our algorithm expedites the consensus decision under best-case conditions (synchrony and no contention) without using data authentication primitives; however, when best-case conditions are not met, data authentication primitives are indeed used.

In the remainder of this section, after presenting the model, we describe our consensus algorithm and then state its optimality. The correctness proof of our consensus algorithm is given in Appendix B.

4.1 Model

We model processes and channels in the same way as in Section 3.1, with the following differences:

- In contrast to Section 3.1: (1) channels are not assumed to be reliable (i.e., messages can be lost), (2) processes that access RQS *can be* Byzantine, and (3) processes that form RQS may directly communicate with each other (this also illustrates application of RQS to different models).
- We assume that the system is eventually synchronous [13] (this is crucial to circumvent the impossibility of an asynchronous consensus [14]). Eventual synchrony means that there is a point in time GST (Global Stabilization Time), not known to processes, such that, after GST, the system is synchronous. In addition, we assume that all messages sent before GST, are either received by GST or lost.
- We allow messages to be authenticated with digital signatures [47]; however, we disallow the use of authenticated messages in best-case executions (to avoid, in best-case executions, the inherent practical latency overhead introduced by signatures). We denote by $\langle m \rangle$ an unauthenticated message, and by $\langle m \rangle_{\sigma_x}$ an authenticated message, i.e., a message signed by process x. We assume that no Byzantine process p_B can forge a digital signature of some benign process p, i.e., if p_B sends $\langle m \rangle_{\sigma_p}$ in execution, ex then p already sent $\langle m \rangle_{\sigma_p}$ in ex.

Our consensus framework is composed of three sets of processes: *proposers*, *acceptors* and *learners* [34]. Roughly, proposers propose values (from domain $\mathbb{D}$) that are to be agreed upon by learners, where the role of acceptors is to help learners agree. In this paper, as in [51], we assume that the set *acceptors* does not intersect with the set *proposers* $\cup$ *learners*, i.e., no proposer or learner can be an acceptor (note that we allow a proposer to be also a learner). We assume that every proposer p is initialized with a single proposal value and all processes are interconnected with point-to-point communication channels.

An algorithm solves consensus if it satisfies the following properties.

– (Validity:) If a benign learner learns a value v and all proposers are benign, then some proposer has proposed v;[4]
– (Agreement:) No two benign learners learn different values;
– (Termination:) If a correct proposer proposes a value, then eventually, every correct learner learns a value.

We construct a refined quorum system $\boldsymbol{RQS}$ around the set $acceptors$ for an adversary $\boldsymbol{B}$, such that $\boldsymbol{RQS}$ is known to all processes. Besides Byzantine acceptors that may belong to adversary, any number of proposers and learners can be Byzantine. Consensus *safety* (i.e., Validity and Agreement) is guaranteed as long as the set of Byzantine acceptors in any execution belongs to $\boldsymbol{B}$, while consensus *liveness* (i.e., Termination) is ensured if there is a correct quorum of acceptors $Q_c \in \boldsymbol{RQS}$.

We say that an execution ex is a *best-case* execution if, in ex: (1) there is no contention, i.e., (a) all proposers are benign and (b) exactly one proposer p proposes, say some value v at time t (and p is correct) and (2) the system is synchronous (during $[t, t+4\Delta]$). Let $\boldsymbol{P}$ be any set of subsets of $acceptors$.

Definition 4. $(m, \boldsymbol{P})$**–fast *consensus algorithm*.** *We say that a consensus algorithm is* $(m, \boldsymbol{P})$– fast *if in every best-case execution ex in which some set* $P \in \boldsymbol{P}$ *contains only correct acceptors, all correct learners learn v in* $m+1$ *message delays*[5]*, without using authenticated messages.*

In the following, we present a novel consensus algorithm, based on RQS, that is $(m, \boldsymbol{QC_m})$*–fast* for all $m \in \{1, 2, 3\}$.

4.2 Consensus Algorithm

Overview. The algorithm consists of two modules: (1) a *Locking* module that ensures safety, and (2) an *Election* module used to help ensure liveness. The *Locking* module consists of a *consult* and an *update* phase.

An execution of the algorithm proceeds in a sequence of *views* (with view numbers taking values from $\mathbb{N}_0$). In every view w, except in the initial view 0 (denoted by $initView$) a single proposer is the *leader*. Leaders are elected by the *Election* module following a round robin fashion (i.e., the leader of view $w \neq initView$ is proposer p_i, where $i = w \bmod |proposers|$). Every proposer p_i is initiated with its proposal value, that it can propose only in $initView$, or in a view in which p_i is the leader.

On proposing a value in a view $w \neq initView$, the leader invokes the *Locking* module. First, p_i initiates the *consult* phase, which, roughly speaking, serves to make sure that p_i changes its proposal value to v_l in case some benign learner learned v_l in some of the previous view. The idea behind the *consult* phase is similar to the view-change subprotocol in the algorithm of Castro and Liskov [7]. The core difference is in the way our algorithm chooses the proposal value in the new view. This is done using $choose()$ function, which we explained later in details. In the *consult* phase, the leader communicates with the acceptor to discover if some value might have been learned. Then, the leader invokes the *update* phase.

[4] The Validity property, as stated in [35], "Only a value proposed by a proposer can be learned", is clearly impossible to ensure in the presence of Byzantine proposers.

[5] In our round-by-round eventually synchronous model [15, 30], a single *message delay* corresponds to a single *round*. In the following, when explaining our algorithm, instead of the term *round*, we use the term *message delay* to prevent confusion with the notion of a round in a sense of a *communication round-trip* used in our storage algorithm (that, in a sense, corresponds to two message delays).

Initialization:
$view, initView := 0$

propose(v) **is** {
  if ($view \neq initView$) **then**
    consult phase
  endif
  update phase }

upon p_j is elected
  propose(v)

Fig. 9. The *Locking* module: High level pseudocode of a proposer p_j

On the other hand, in $initView$ all proposers can be seen as leaders. As shown in Figure 9, which gives the high-level pseudocode of the *Locking* module, in $initView$, the proposer, on proposing a value, skips the *consult* phase and executes directly the *update* phase.

Communication pattern of the *update* phase is illustrated in Figure 10; it takes 4 communication steps and allows correct learners to learn a value in $m+1$ communication steps in best-case executions in which there is a quorum of class m which contains only correct acceptors (for $m \in \{1, 2, 3\}$). In the first round of this phase, called prepare round, proposer communicates with acceptors. This is then followed by 3 update rounds, in which acceptors send messages to all acceptors and learners. Learners can learn a value at the end of any of the update rounds (i.e., rounds 2, 3 or 4).

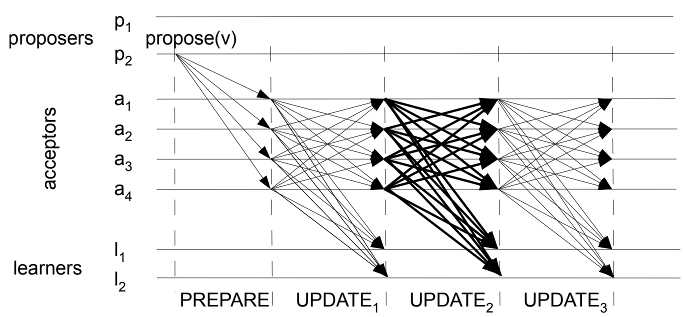


Fig. 10. Communication pattern of the *update* phase. An acceptor may send multiple update$_2$ messages (bolded).

In the following, we focus on the *Locking* module. We first explain the *update* phase (given in Fig. 11) since it is the only part of the algorithm involved in a best-case execution. Then, we detail the *consult* phase (Fig. 12) with the particular emphasis on the *choose*() function (Fig. 13). Finally, we give a simple *Election* module (Fig. 14). The correctness proof of our consensus algorithm is postponed to Appendix B.

Notice that the complete pseudocode of the *Locking* module, combined from pseudocodes of Figures 9, 11 and 12 is given in Figure 15. The additional part of the *Locking* module consists of lines 40 and 101-104, Fig. 15, that serve solely to halt a consensus instance, i.e., to permanently stop view changes, i.e., to halt the *Election* module. Otherwise, lines 40 and 101-104 of Fig. 15 can be omitted.

---

at every proposer p_j**:**
Initialization:
$view, initView := 0$; $vProof := nil$; $Q := \emptyset$
9: send $\mathsf{prepare}\langle v, view, vProof, Q\rangle$ to *acceptors*

---

at every acceptor a_j**:**
Initialization:
$view_{a_j} := initView$; $Prep_{view}, old, Update_{proof}[*,*], Update_{view}[*], Update_Q[*,*] := \emptyset$; $Prep, Update[*] := nil$

upon received $m = \mathsf{prepare}\langle v, view_{a_j}, vProof, Q\rangle$ from p_i
31: **if** $(w \in Prep_{view} \Rightarrow w < view_{a_j})$ **and** ($view_{a_j} = initView$ **or** (p_i is leader **and** v matches **choose**$(v, vProof, Q)$)) **then**
32: **if** $Prep = v$ **then** $Prep_{view} := Prep_{view} \cup \{view_{a_j}\}$ **else** $Prep := v$; $Prep_{view} := \{view_{a_j}\}$
33: send $m_1 = \mathsf{update}_1\langle v, view_{a_j}, \emptyset\rangle$ to $acceptors \cup learners$; $old := old \cup m_1$

upon received $m = \mathsf{update}_{step}\langle v, view_{a_j}, *\rangle$ from some quorum Q **and** $v = Prep$ **and** $view_{a_j} \in Prep_{view}$ (for $step \in \{1, 2\}$)
34: **if** $Update[step] = v$ **then** $Update_{view}[step] := Update_{view}[step] \cup \{view_{a_j}\}$
35: **else** $Update[step] := v$; $Update_{view}[step] := \{view_{a_j}\}$; $Update_Q[step, *] := \emptyset$; $Update_{proof}[step, *] := \emptyset$
36: **if** ($Q \notin Update_Q[step, view_{a_j}]$ **and** $step = 1$) **or** ($Update_Q[step, view_{a_j}] = \emptyset$ **and** $step = 2$) **then**
37: $Update_Q[step, view_{a_j}] := Update_Q[step, view_{a_j}] \cup Q$
38: send $m_{step+1} = \mathsf{update}_{step+1}\langle v, view_{a_j}, Q\rangle$ to $acceptors \cup learners$; $old := old \cup m_{step+1}$

---

at every acceptor and learner x**:**
upon received the same $\mathsf{update}_1\langle v, view, *\rangle$ from $Q_1 \in \boldsymbol{QC_1}$
51: **if** x has not yet decided **then decide**(v)

upon received the same $\mathsf{update}_2\langle v, view, Q_2\rangle$ from $Q_2 \in \boldsymbol{QC_2}$
52: **if** x has not yet decided **then decide**(v)

upon received the same $\mathsf{update}_3\langle v, view, *\rangle$ from $Q_3 \in \boldsymbol{RQS}$
53: **if** x has not yet decided **then decide**(v)

at every learner l_j**:**
upon l_j decides v
60: **learn**(v)

Fig. 11. The *Locking* module: *update* phase

***Update* phase.** The 4 communication steps of the *update* phase proceed as follows (line numbers refer to Fig. 11):

1. Proposer p sends a message $m = \mathsf{prepare}$ to all acceptors (line 9) containing: (a) its proposal value v, (b) the view number $view$, and (c) the array of authenticated messages $vProof$ that originates from some quorum Q of acceptors. Roughly, $vProof$ serves as a certificate for the proposed value v. We detail how $vProof$ is constructed later when explaining the *consult* phase. It is important to notice that, in the $initView$, $vProof$ equals nil (i.e., contains no messages).
2. Benign acceptor a_j, upon receiving $m = \mathsf{prepare}\langle v, view, vProof, Q\rangle$ from p, such that $view = view_{a_j}$ (i.e., if a_j is in $view$), checks if (line 31): (a) (unless $view = initView$) whether p is the leader of $view$ and whether $vProof$ matches value v (this is done using the *choose*() function that is explained later in details), and (b) a_j did not already receive a $\mathsf{prepare}$ message in $view$. If these checks succeed, a_j stores v into a local variable $Prep$ and the view number in the set variable $Prep_{view}$ (we simply say, a_j *prepares* v in $view$). If $Prep$ was already equal to v, then a_j simply adds $view$ to the set $Prep_{view}$ (line 32). Then a_j echoes v by sending an $\mathsf{update}_1\langle v, view, \emptyset\rangle$ message to all acceptors and learners (line 33).

3. Benign acceptor a_j, upon receiving update_1 messages from some quorum Q with the same value v and view number *view*, checks if its local view equals *view* and if it already prepared a message with a value v in *view*. If this check succeeds, a_j performs the following local computations (we say a_j *1-updates* v in *view* with quorum Q). In case the local variable $Update[1]$ does not equal v (i.e., if a new value is 1-updated — line 35), a_j: (i) stores v into $Update[1]$ and *view* into $Update_{view}[1]$, and (ii) empties the sets $Update_Q[1,*]$ and $Update_{proof}[1,*]$. In case a value v was already 1-updated (in some previous view — line 34), a_j simply adds *view* into $Update_{view}[1]$. Then, a_j adds the identifier of the quorum Q into the set $Update_Q[1, view]$ and sends an $\mathsf{update}_2\langle v, view, Q\rangle$ message to all acceptors and learners (here, an update_2 message is sent once per every different quorum Q — line 38).
4. A benign acceptor a_j, upon receiving update_2 message from some quorum Q, performs the similar steps as when receiving a quorum of update_1 messages (we say a_j *2-updates* v in *view*), including sending an update_3 message containing v and *view* to all acceptors and learners (lines 34-38). The differences with respect to the step (3) are captured in lines 36-38; namely in step (4) an acceptor: (i) adds only one quorum Q (the first one) to $Update_Q[2,*]$ per view (lines 36 and 37), and (ii) sends only one $\mathsf{update}_3\langle v, view, *\rangle$ message per view to other acceptors and learners.

Moreover, all acceptors and learners *decide* on v upon receiving update_1 messages with the same value v and view number *view* from a class 1 quorum (line 51). Similarly, acceptors and learners decide on v upon receiving the same $\mathsf{update}_2\langle v, view, Q_2\rangle$ messages from some class 2 quorum Q_2 (note here that, besides value and the view number, the quorum identifier within update_2 messages must be the same — line 52). Finally, acceptors and learners decide on v upon receiving update_3 messages with the v and *view* from any (class 3) quorum of acceptors (line 53). Besides, a benign learner l_j learns v as soon as l_j decides v (line 60).

The above scheme guarantees that, in the best case execution, in which only a single proposer proposes in the $initView$ and the system is synchronous, all correct learners learn v in two (resp., three; four) message delays in case a class 1 (resp., class 2; class 3) quorum of correct servers is available.[6] Note that, in the above sequence, all messages are unauthenticated.

***Consult* phase.** In a best-case execution, the *Election* module, responsible for view changes, does not change the view before all correct acceptors (and learners) decide v. However, if more than one proposer proposes in $initView$, or some proposer is Byzantine, or if the system is asynchronous, the *Election* module might designate a different proposer p_i to be the leader for the new view w (see Fig. 9) which then invokes the *consult* phase of the *Locking* module.

Proposer p_i starts the *consult* phase of a new view w by sending the $\mathsf{new\_view}$ message to acceptors (line 2, Fig. 12). The $\mathsf{new\_view}$ message contains a view number and a set of messages, $viewProof$, which are provided to the proposer by the *Election* module. The set $viewProof$ consists of signed (authenticated) messages from a quorum of acceptors — this vouches for the authenticity of the $\mathsf{new\_view}$ message. After sending the $\mathsf{new\_view}$ message, p_i waits for a quorum Q of *valid* signed acks (line 4) containing the last prepared, 1-updated and 2-updated values, along with the corresponding view numbers. An acceptor a_j acks a $\mathsf{new\_view}$ message only if (line 21, Fig. 12): (a) the view number w is higher than the acceptor's local view number $view_{a_j}$, (b) p_i is the leader of

[6] Since an availability of a class 3 quorum is anyway assumed, our algorithm guarantees that a value will be learned by all correct learners in at most four message delays in any best-case execution.

the view w (i.e., if $i = w \,\mathsf{mod}\, |proposers|$), and (c) the set $viewProof$ matches w, i.e., if $viewProof$ proof contains a quorum of authenticated messages vouching that p_i may issue a new_view message for a view w.

---

at every proposer p_j**:**
Initialization:
$view, initView := 0$; $viewProof, vProof := nil$; $\boldsymbol{faulty} := \emptyset$
2: send new_view$\langle view, viewProof\rangle$ to $acceptors$
3: **repeat**
4: **wait for** valid acks from some quorum $Q \in \boldsymbol{RQS} \setminus \boldsymbol{faulty}$
5: $vProof :=$ array of received acks from Q
6: $(v, abort) :=$ **choose**$(v, vProof, Q)$
7: **if** $abort$ **then** $\boldsymbol{faulty} := \boldsymbol{faulty} \cup \{Q\}$
8: **until** $\neg(abort)$

---

at every acceptor a_j**:**
Initialization:
$view_{a_j} := initView$; $Prep_{view}, old, Update_{proof}[*,*], Update_{view}[*], Update_Q[*,*] := \emptyset$; $Prep, Update[*] := nil$

upon received new_view$\langle view, viewProof\rangle$ from p_i
21: **if** $(view > view_{a_j})$ **and** (p_i is the leader of $view$) **and** ($viewProof$ matches $view$) **then**
22: $view_{a_j} := view$
23: $\forall step \in \{1,2\}, \forall w : w \in Update_{view}[step] \wedge Update_{proof}[step, w] = \emptyset$ **do**
24: send sign_req$\langle Update[step], w, step\rangle$ to some quorum in $Update_Q[step, w]$
25: **for** every sent sign_req$\langle Update[step], w, step\rangle$ message
26: **wait for** acks with a valid signature from some subset of $acceptors$ $T_{step,w}$, $T_{step,w} \notin \boldsymbol{B}$
27: $Update_{proof}[step, w] :=$ received acks from $T_{step,w}$
28: send new_view_ack$\langle view_{a_j}, Prep, Prep_{view}, Update[1..2], Update_{view}[1..2], Update_{proof}[1..2,*], Update_Q[1..2,*]\rangle_{\sigma_{a_j}}$ to p_i

upon received sign_req$\langle v, w, step\rangle$ from a_i
29: **if** $m =$ update$_{step}\langle v, w, *\rangle \in old$ **then** send sign_ack$\langle m\rangle_{\sigma_{a_j}}$ to a_i

---

Fig. 12. The *Locking* module: *consult* phase

An ack (i.e., a new_view_ack message — line 28, Fig. 12) for a view w is considered *valid* in line 4, Fig. 12 if every value v_{step} in $Update[step]$ and every view number w' in $Update_{view}[step]$ ($step \in \{1,2\}$) is accompanied by a set of signatures, $Update_{proof}[step, w']$. Here, every $Update_{proof}[step, w']$ must be a set of signed update$_{step}\langle v_{step}, w', *\rangle$ messages sent from all acceptors from some subset of acceptors that is not an element of an adversary (to guarantee that a message is signed by at least one benign acceptor). An acceptor a_j must obtain all the necessary sets of signatures before replying to the new_view message — this is done in lines 23-27 and 29, Fig. 12, unless a_j already possesses the required proofs. (Notice here that the variable *old* used in line 29, Fig. 12 is the same variable *old* from Fig. 11.)

Then, the leader of view w, p_i, evaluates acks from Q using the *choose*() function (line 6, Fig. 12 and Fig. 13).

***Choose* function.** The *choose*() function is the heart of our algorithm and relies on RQS properties to guarantee consensus safety. This function ensures the following crucial property: *if any value v is decided in a view w, then benign acceptors in a view higher than w accept only v*. We sketch the arguments (based on RQS properties) behind this property, for the view $w+1$ (which gives the base step of the induction-based proof). In the following, we refer to Figure 13.

Definitions:

1: $Cand_2(v, w, Q) ::= \exists Q_1 \in \boldsymbol{QC_1}, \exists B \in \boldsymbol{B}, \forall a_j \in (Q_1 \cap Q) \setminus B : (w \in vProof[a_j].Prep_{view}) \wedge (vProof[a_j].Prep = v)$

2: $C_3(v, w, char, Q_2, B, Q) ::= \forall a_j \in (Q_2 \cap Q) \setminus B :$
$\quad P3_{char}(Q_2, Q, B) \wedge (vProof[a_j].Update[1] = v) \wedge (w \in vProof[a_j].Update_{view}[1]) \wedge (Q_2 \in vProof[a_j].Update_Q[1, w])$
3: $Cand_3(v, w, char, Q) ::= \exists Q_2 \in \boldsymbol{QC_2}, \exists B \in \boldsymbol{B}$: $C_3(v, w, char, Q_2, B, Q)$
4: $Valid_3(v, w, char, Q) ::= \forall Q_2 \in \boldsymbol{QC_2}, \forall B \in \boldsymbol{B}, \forall a_j \in Q_2 \cap Q, \forall w' \in \mathbb{N} :$
$\quad C_3(v, w, char, Q_2, B, Q) \Rightarrow ((vProof[a_j].Prep = v) \wedge (w \in vProof[a_j].Prep_{view})) \vee (w' \in vProof[a_j].Prep_{view} \Rightarrow w' > w)$

5: $Cand_4(v, w, Q) ::= \exists a_j \in Q : (vProof[a_j].Update[2] = v) \wedge (w \in vProof[a_j].Update_{view}[2])$

choose$(v', vProof, Q)$ **returns**$(v_{ret}, abort)$ **is** {
10: $v_{ret} := v'$; $abort := false$
11: **if** $\exists v \in \mathbb{D}, \exists w \in \mathbb{N}_0, \exists char \in \{‘a‘, ‘b‘\} : Cand_2(v, w, Q) \vee Cand_3(v, w, char, Q) \vee Cand_4(v, w, Q)$ **then**
12: $\quad view_{max} := max\{w \in \mathbb{N}_0 \mid \exists v \in \mathbb{D}, \exists char \in \{‘a‘, ‘b‘\} : Cand_2(v, w, Q) \vee Cand_3(v, w, char, Q) \vee Cand_4(v, w, Q)\}$
13: $\quad$ **if** $\exists v \in \mathbb{D} : Cand_3(v, view_{max}, ‘a‘, Q) \vee Cand_4(v, view_{max}, Q)$ **then**
14: $\quad\quad v_{ret} := v$; **return**
15: $\quad$ **if** $\exists v, v' \in \mathbb{D} : (v \neq v') \wedge Cand_3(v, view_{max}, ‘b‘, Q) \wedge Cand_3(v', view_{max}, ‘b‘, Q)$ **then**
16: $\quad\quad abort := true$; **return**
17: $\quad$ **if** $\exists v \in \mathbb{D} : Cand_3(v, view_{max}, ‘b‘, Q)$ **then**
18: $\quad\quad$ **if** $Valid_3(v, view_{max}, ‘b‘, Q)$ **then** $v_{ret} := v$ **else** $abort := true$
19: $\quad\quad$ **return**
20: $\quad v_{ret} := v \in \mathbb{D} : Cand_2(v, view_{max}, Q)$; **return**
21: **else return**

Fig. 13. *choose()* function

Let v be the value decided by some benign process (acceptor or learner) in view w upon receiving update_k messages from some class k quorum Q_k. Then, for any Q, substituting for Q_1 (resp., Q_2; Q_3), $Cand_2(v, w, Q)$ holds in line 1 (resp., $Cand_3(v, w, char, Q)$ holds for some $char \in \{‘a‘, ‘b‘\}$ in line 3; $Cand_4(v, w, Q)$ holds in line 5). In this case, we say that v is a *candidate* with view number w. It is not difficult to see that there can be no candidate v with view $w' > w$ (since no benign acceptor prepares or updates any value in a view higher than w), i.e., $w = view_{max}$ in line 12. Hence, $choose()$ may return only the candidate with view number w. However, $choose()$ faces the challenging tasks of prioritizing different candidate values with the same view number w. In the following, we illustrate how $choose()$ relies on RQS properties to safely prioritize candidate values, by focusing a particular example in which v was decided in view w after some benign acceptor or learner receives update_1 messages from some class 1 quorum Q_1. In this case, since v was decided, $choose()$ must not return a value $v' \neq v$. We show that, for a valid $vProof$ that consists of new_view_ack messages sent by *any* quorum Q, $choose(*, vProof, Q)$ either returns v or sets the *abort* flag. In the latter case, we will show that Q actually contains some Byzantine acceptor; in this case, the proposer will simply wait for additional new_view_ack messages from other acceptors and re-invoke $choose()$ until it encounters a quorum that does not contain Byzantine acceptors, when $choose()$ will not abort (see lines 3-8, Fig. 12).

As we suggested above, in this case, $Cand_2(v, w, Q)$ holds. To see this, denote the set of Byzantine acceptors in any execution ex by $B_{ex} \in \boldsymbol{B}$. Then $vProof$ contains new_view_ack messages from all acceptors from the set $X = Q_1 \cap Q \setminus B_{ex}$. Since all acceptors from X are benign they all correctly inform the proposer that they prepared v in view w and, hence, $Cand_2(v, w, Q)$ holds. The following arguments show that the low priority that candidate values for which $Cand_2(v, w, Q)$ holds have in $choose()$ (such candidate values can be returned only in line 20), does not compromise safety:

1. If, for value, $v' \neq v$: (i) $Cand_3(v', w, ‘a‘, Q)$ or (ii) $Cand_4(v, w, Q)$ hold, v' will be returned in line 14. However, this is not possible. In case (i), there are class 2 quorum Q_2 and a set $B \in \boldsymbol{B}$ such

that all acceptors from $Y = Q_2 \cap Q \setminus B$ reported they 1-updated v' in w. Since, $P_{3a}(Q_2, Q, B)$ holds, this includes at least one benign acceptor. Similarly, in case (ii) some acceptor $a_j \in Q$ reported that it 2-updated v' in w. This means that in case (ii), just like in case (i), there is at least one benign acceptor that 1-updated v' in w (in a valid $vProof$, $vProof[a_j].Update_{proof}[2]$ contains a set of signatures from all acceptors from the set $T \notin \boldsymbol{B}$ confirming that they 1-updated v' in w). Since benign acceptors 1-update v in w only if all acceptors from $Q' \setminus B_{ex}$ (for some quorum Q') prepared v in w — by Property 1 of RQS, since $Q_1 \cap Q' \setminus B_{ex}$ is non-empty, this would imply that at least one benign acceptor prepared both v and v' in w. A contradiction.

2. It is less obvious that $choose()$ cannot return $v' \neq v$ in line 18, if both $Cand_3(v', w, `b`, Q)$ and $Valid_3(v', w, `b`, Q)$ hold. However, this is also impossible. Namely, in this case, there is a class 2 quorum Q_2 and $B \in \boldsymbol{B}$, such that all acceptors from $Y = Q_2 \cap Q \setminus B$ claim they 1-updated v' in w. If $Y \nsubseteq B_{ex}$ there is one benign acceptor that indeed 1-updated v' in w — a contradiction follows the argument in the previous paragraph. Otherwise, if $Y \subseteq B_{ex}$, since $Valid_3(v', w, `b`, Q)$ holds, all (benign) acceptors from $Z = Q_2 \cap Q \setminus B_{ex}$ prepared v' in w.[7] If $P_{3b}(Q_2', Q, B_{ex})$ holds, this implies that $Z \cap Q_1$ is non-empty, i.e., at least one benign acceptor prepared both v and v' in w — a contradiction. The last possibility, that $P_{3a}(Q_2, Q, B_{ex})$ holds, implies that $P_{3a}(Q_2, Q, Y)$ also holds (since $Y \subseteq B_{ex}$). However, this implies $Q_2 \cap Q \setminus Y \notin \boldsymbol{B}$, which contradicts the definition of Y.
3. Finally, it should not be difficult to see, by applying Property 2 of RQS, that $Cand_2(v', w, Q)$ cannot hold at the same time as $Cand_2(v, w, Q)$, for some $v' \neq v$. Hence, choosing a candidate value in line 20 is not ambiguous.

Finally, in the following we explain why $choose(*, vProof, Q)$ never aborts in case quorum Q contains only benign acceptors. Assume, by contradiction, that $choose(*, vProof, Q)$ aborts, yet Q contains only benign acceptors.

1. If $choose()$ aborts in line 16, there are two values v and $v' \neq v$ such that both $Cand_3(v, w, `b`, Q)$ and $Cand_3(v', w, `b`, Q)$ hold. In this case there are (supposedly benign) acceptors $a_i, a_j \in Q$ such that a_i (resp., a_j) claims it 1-updated v (resp., v') in w, i.e., that it received $\mathsf{update}_1\langle v, w, \emptyset\rangle$ ($\mathsf{update}_1\langle v', w, \emptyset\rangle$) messages from some quorum Q' (resp., Q'') of acceptors. By Property 1 of RQS, $Q' \cap Q'' \setminus B_{ex} \neq \emptyset$, i.e., there is at least one benign acceptor a_k that sent both $\mathsf{update}_1\langle v, w, \emptyset\rangle$ and $\mathsf{update}_1\langle v', w, \emptyset\rangle$, i.e., a_k prepared both v and v' in view w. A contradiction.
2. If $choose()$ aborts in line 18 then, there is a value v such that $Cand_3(v, w, `b`, Q)$ holds, yet $Valid_3(v, w, `b`, Q)$ does not hold. In this case, there is a class 2 quorum Q_2 and a (benign) acceptor $a_i \in Q$ that claims it received $\mathsf{update}_1\langle v, w, \emptyset\rangle$ messages from all (supposedly benign) acceptors from $Q_2 \cap Q$, i.e., that all acceptors from $Q_2 \cap Q$ prepared v in w. However, since $Valid_3(v, w, `b`, Q)$ does not hold, there is a benign acceptor $a_j \in Q_2 \cap Q$ for which the predicate $P = P_1 \vee P_2$, such that:

$$P_1 ::= (vProof[a_j].Prep = v) \wedge (w \in vProof[a_j].Prep_{view}), \text{ and}$$
$$P_2 ::= w' \in vProof[a_j].Prep_{view} \Rightarrow w' > w.$$

does not hold. Since a_j prepared v in w, there are two possibilities.

[7] Acceptors from Z could not have prepared a value in a view higher than w to satisfy $Valid_3(v', w, `b`, Q)$ since we consider only $vProof$ for $w+1$ in this example. Benign acceptors must be in the view lower than the one for which they send a new_view_ack message.

(a) a_j never prepared a value different than v in views higher than w; in this case P_1 must hold, since a_j never removes w from its variable $Prep_{view}$ (line 32, Fig. 11) — a contradiction.

(b) a_j prepared a value different from v in a view higher than w, say in view w'. Then (line 32, Fig. 11), $Prep_{view}$ at a_j contains only view numbers higher than w, i.e., P_2 must hold — a contradiction.

The *Election* module. The Election module given in Figure 14 is very simple and guarantees progress in case the system is eventually synchronous. It is based on an exponential increase of the timeouts (maintained by acceptors) between views. This scheme can be seen as inefficient, and impact the worst-case performance of our algorithm. Different optimizations of this simple scheme are possible, but these are out of the scope of this paper.

In the following we state the optimality of our algorithm.

---

at every acceptor a_j:
$suspectTimeout, initTimeout := 5\Delta$; $nextView_{a_j} := initView$ % Initialization

upon reception of prepare$\langle *, initView, *, * \rangle$ or sync message for the first time
  trigger($suspectTimeout$)

upon expiration of ($suspectTimeout$)
  $suspectTimeout := suspectTimeout * 2$
  inc($nextView_{a_j}$)
  $nextLeader := nextView_{a_j}$ mod $|proposers|$
  send view_change$\langle nextView_{a_j} \rangle_{\sigma_{a_j}}$ to $p_{nextLeader}$
  trigger($suspectTimeout$)

upon decide(v)
  send decision$\langle v \rangle$ to $acceptors$

upon reception of a valid decision$\langle v \rangle$ from some quorum $Q \in \boldsymbol{RQS}$
  stop($suspectTimeout$)

at every proposer p_j:

upon reception of view_change$\langle nextView \rangle_{\sigma_{a_i}}$ with the same $nextView$ from all a_i from some $Q \in \boldsymbol{RQS}$
  if $nextView > view$ **then**
    $viewProof := \cup$ received signed view_change$\langle nextView \rangle$ messages
    $view := nextView$
    elect($self$)

upon p_j proposed a value for the first time
  wait some preset time
  send sync to $acceptors$
  send $\langle$decision_pull$\rangle$ to $acceptors$

upon received decision$\langle v \rangle$ (with the same v) from some quorum $Q \in \boldsymbol{RQS}$
  halt

Fig. 14. The *Election* module

at every proposer p_j:
Initialization:
$view, initView := 0$; $viewProof, vProof := nil$; Q, $\boldsymbol{faulty} := \emptyset$
propose(v) **is** {
1: **if** $(view \neq initView)$ **then** % *consult* phase
2: send new_view$\langle view, viewProof\rangle$ to *acceptors*
3: **repeat**
4: **wait for** valid acks from some quorum $Q \in \boldsymbol{RQS} \setminus \boldsymbol{faulty}$
5: $vProof$:= array of received acks from Q
6: $(v, abort) :=$ **choose**$(v, vProof, Q)$
7: **if** $abort$ **then** $\boldsymbol{faulty} := \boldsymbol{faulty} \cup \{Q\}$
8: **until** $\neg(abort)$
9: send prepare$\langle v, view, vProof, Q\rangle$ to *acceptors* % *update* phase

upon p_j is elected
10: **propose**(v)

---

at every acceptor a_j:
Initialization:
$view_{a_j} := initView$; $Prep_{view}, old, Update_{proof}[*,*], Update_{view}[*], Update_Q[*,*] := \emptyset$; $Prep, Update[*] := nil$

upon received new_view$\langle view, viewProof\rangle$ from p_i % *consult* phase (lines 21-29)
21: **if** $(view > view_{a_j})$ **and** (p_i is the leader of $view$) **and** ($viewProof$ matches $view$) **then**
22: $view_{a_j} := view$
23: $\forall step \in \{1,2\}, \forall w : w \in Update_{view}[step] \wedge Update_{proof}[step, w] = \emptyset$ **do**
24: send sign_req$\langle Update[step], w, step\rangle$ to some quorum in $Update_Q[step, w]$
25: **for** every sent sign_req$\langle Update[step], w, step]\rangle$ message
26: **wait for** acks with a valid signature from some subset of *acceptors* $T_{step,w}$, $T_{step,w} \notin \boldsymbol{B}$
27: $Update_{proof}[step, w]$:= received acks from $T_{step,w}$
28: send new_view_ack$\langle view_{a_j}, Prep, Prep_{view}, Update[1..2], Update_{view}[1..2], Update_{proof}[1..2,*], Update_Q[1..2,*]\rangle_{\sigma_{a_j}}$ to p_i

upon received sign_req$\langle v, w, step\rangle$ from a_i
29: **if** $m =$ update$_{step}\langle v, w, *\rangle \in old$ **then** send sign_ack$\langle m\rangle_{\sigma_{a_j}}$ to a_i

upon received $m =$ prepare$\langle v, view_{a_j}, vProof, Q\rangle$ from p_i % *update* phase (lines 31-38 and 51-60)
31: **if** $(w \in Prep_{view} \Rightarrow w < view_{a_j})$ **and** $(view_{a_j} = initView$ **or** (p_i is leader **and** v matches **choose**$(v, vProof, Q)$)) **then**
32: **if** $Prep = v$ **then** $Prep_{view} := Prep_{view} \cup \{view_{a_j}\}$ **else** $Prep := v$; $Prep_{view} := \{view_{a_j}\}$
33: send $m_1 =$ update$_1\langle v, view_{a_j}, \emptyset\rangle$ to $acceptors \cup learners$; $old := old \cup m_1$

upon received $m =$ update$_{step}\langle v, view_{a_j}, *\rangle$ from some quorum Q **and** $v = Prep$ **and** $view_{a_j} \in Prep_{view}$ (for $step \in \{1,2\}$)
34: **if** $Update[step] = v$ **then** $Update_{view}[step] := Update_{view}[step] \cup \{view_{a_j}\}$
35: **else** $Update[step] := v$; $Update_{view}[step] := \{view_{a_j}\}$; $Update_Q[step,*] := \emptyset$; $Update_{proof}[step,*] := \emptyset$
36: **if** $(Q \notin Update_Q[step, view_{a_j}]$ **and** $step = 1)$ **or** $(Update_Q[step, view_{a_j}] = \emptyset$ **and** $step = 2)$ **then**
37: $Update_Q[step, view_{a_j}] := Update_Q[step, view_{a_j}] \cup Q$
38: send $m_{step+1} =$ update$_{step+1}\langle v, view_{a_j}, Q\rangle$ to $acceptors \cup learners$; $old := old \cup m_{step+1}$

upon reception of $\langle$decision_pull$\rangle$ from a process q
40: **if** decided v **then** send decision$\langle v\rangle$ to $acceptors \cup \{q\}$

---

at every acceptor and learner x:
upon received the same update$_1\langle v, view, *\rangle$ from $Q_1 \in \boldsymbol{QC_1}$
51: **if** x has not yet decided **then decide**(v)

upon received the same update$_2\langle v, view, Q_2\rangle$ from $Q_2 \in \boldsymbol{QC_2}$
52: **if** x has not yet decided **then decide**(v)

upon received the same update$_3\langle v, view, *\rangle$ from $Q_3 \in \boldsymbol{RQS}$
53: **if** x has not yet decided **then decide**(v)

at every learner l_j:
upon l_j decides v
60: **learn**(v)

---

at every learner l_j:
upon l_j received decision$\langle v\rangle$ from some subset of *acceptors* T, $T \notin \boldsymbol{B}$
101: **if** l_j has not yet learned a value **then learn**(v)

upon value not learned
102: **wait** some preset time
103: **if** value not learned **then** send $\langle$decision_pull$\rangle$ to *acceptors*

Fig. 15. The *Locking* module

4.3 Optimality

We say that an algorithm A implements $(\boldsymbol{Q},\boldsymbol{B})$*–consensus* if A ensures consensus Validity and Agreement, as long as, for any execution ex of A, the set of acceptors Byzantine in ex belongs to $\boldsymbol{B}$, as well as Termination in case the system is eventually synchronous and there is a set $Q \in \boldsymbol{Q}$ that contains only correct acceptors. Denoting by $\boldsymbol{Q^{(i)}}$ $(i = 1 \dots 3)$ some set of subsets of *acceptors*, the following theorems capture the minimality of our RQS, assuming $|proposers| \geq 2$ and $|learners| \geq 3$.[8]

Theorem 4. *If an algorithm A implements $(\boldsymbol{Q^{(3)}},\boldsymbol{B})$*–consensus*, then $P1(\boldsymbol{Q^{(3)}},\boldsymbol{B})$ holds.*

Theorem 5. *If a $(\boldsymbol{Q^{(3)}},\boldsymbol{B})$*–consensus *algorithm A is* $(1,\boldsymbol{Q^{(1)}})$–fast*, then $P2(\boldsymbol{Q^{(1)}},\boldsymbol{Q^{(3)}},\boldsymbol{B})$ holds.*

Theorem 6. *If a $(\boldsymbol{Q^{(3)}},\boldsymbol{B})$*–consensus *algorithm A is both* $(1,\boldsymbol{Q^{(1)}})$–fast *(for some $\boldsymbol{Q^{(1)}} \neq \emptyset$) and* $(2,\boldsymbol{Q^{(2)}})$–fast*, then $P3(\boldsymbol{Q^{(1)}},\boldsymbol{Q^{(2)}},\boldsymbol{Q^{(3)}},\boldsymbol{B})$ holds.*

Theorem 4 is not new; it follows directly from [29]. Moreover, in the special threshold case, where (a) $\boldsymbol{B}{=}\boldsymbol{B^k}$, (b) all elements of $\boldsymbol{Q^{(1)}}$ (resp., $\boldsymbol{Q^{(3)}}$) contain at least $n-q$ (resp., $n-t$) acceptors (where n denotes the total number of acceptors), and (c) $q = t - 2k$, Theorems 4–5 correspond to the lower bounds identified in [35].

In the following, we prove Theorem 6. To strengthen the optimality result established by Theorem 6 we assume that proposers and learners may not be Byzantine, yet that any number of proposers and learners may fail by crashing.

Proof. **Preliminaries.** To precisely prove Theorem 6, we assume full information protocols in the *round-by-round* eventually synchronous model [15, 30]. The assumption of a full information protocol is indeed without loss of generality, since if, in some particular algorithm A, a process p does not send a message to the process q in round rnd, we model this by having process p send to q a default message msg_{nil} in rnd and q does not change its state upon reception of a message msg_{nil}. Moreover, denote by $m_i^j.p[q]$ the message sent by process p to process q in round j of some execution ex_i. For presentation simplicity we assume that, in each round, every process combines all the messages $m_i^j.p[q]$ it is about to send in round j and sends the same message $m_i^j.p$ to all processes, such that every process q (including Byzantine ones) ignores all the portions of the message except $m_i^j.p[q]$ (it is not difficult to see that this is indeed without loss of generality). Finally, we denote by $M_i^j.X$ the set of all messages $m_i^j.p$, where $p \in X$ (i.e., $M_i^j.X = \{m_i^j.p | p \in X\}$).

Assume, by contradiction, that there is a $(\boldsymbol{Q^{(3)}},\boldsymbol{B})$*–consensus* algorithm A that is both $(1,\boldsymbol{Q^{(1)}})$*–fast* (for some $\boldsymbol{Q^{(1)}} \neq \emptyset$) and $(2,\boldsymbol{Q^{(2)}})$*–fast* such that $P3(\boldsymbol{Q^{(1)}},\boldsymbol{Q^{(2)}},\boldsymbol{Q^{(3)}})$ is violated, i.e.:

$$\exists Q_1 \in \boldsymbol{Q^{(1)}},\ \exists Q_2 \in \boldsymbol{Q^{(2)}},\ \exists Q \in \boldsymbol{Q^{(3)}},\ \exists B_1', B_2 \in \boldsymbol{B}\colon (Q_2 \cap Q \setminus B_1' = B_2) \wedge (Q_1 \cap Q_2 \cap Q \subseteq B_1').$$

In the following, we denote the set $Q_1 \cap Q_2 \cap Q$ by B_0 and $Q_2 \cap Q \cap B_1'$ by B_1. Having in mind that $\boldsymbol{B}$ is an adversary for S, it is straightforward to see that (i) $B_0, B_1 \subseteq B_1'$, (ii) $B_0, B_1 \in \boldsymbol{B}$, and (iii) $Q_2 \cap Q = B_1 \cup B_2$. Moreover, since $B_0 \subseteq B_1'$ and $B_0 \subseteq Q_2 \cap Q$, we have $B_0 \subseteq B_1$. Furthermore, denote by $\overline{X}$ the set $acceptors \setminus X$, where X is any subset of $acceptors$. Hence, $Q_2 \cap Q \cap \overline{Q_1} = B_2 \cup (B_1 \setminus B_0)$.

[8] We exclude here the special cases where $|proposers| = 1$, $|learners| \leq 2$ or $acceptors \cap (proposers \cup learners) \neq \emptyset$). These have to be addressed separately.

Denote by p_0 and p_1 two distinct proposers ($p_0 \neq p_1$) (such proposers exist since $|proposers| \geq 2$). Since there are at least three learners distinct learners (because $|learners| \geq 3$), there is a learner in $learners \setminus \{p_0, p_1\}$ — we denote this learner by l_2. Moreover, there are two different learners distinct from l_2: we denote these by l_0 and l_1. Without loss of generality, if $p_0 \in \{l_0, l_1\}$ (resp., $p_1 \in \{l_0, l_1\}$), we assume $p_0 = l_0$ (resp., $p_1 = l_1$). Recall here that $(proposers \cup learners) \cap acceptors = \emptyset$.

We only consider the cases where p_0 proposes 0 and p_1 proposes 1 (as this is sufficient to prove the theorem). Let $m0 = m_i^1.p_0$ (resp., $m1 = m_i^1.p_1$) be the message sent by p_0 (resp., p_1) in round 1 of some ex_i, when p_0 (resp., p_1) is correct and proposes 0 (resp., 1) at the beginning of round 1 of ex_i (notice that we consider deterministic algorithms so $m0$ and $m1$ do not depend on a given ex_i). We say that a process a_i *plays* 0 (resp. 1) to some process a_j in round 2 of ex_i if a_j cannot distinguish, at round 2, execution ex_i from some execution ex' in which (1) a_i has received $m0$ (resp. $m1$) from p_0 (resp., p_1) in the first round, and (2) a_i is correct.

To exhibit a contradiction, we construct several (partial) executions of the algorithm A, including the one in which *agreement* is violated. In these executions, we consider only processes belonging to the set $acceptors \cup \{p_0, p_1, l_0, l_1, l_2\}$. Other processes can be assumed w.l.o.g. to fail by crashing at the beginning of each of the following executions.

$\underline{ex_0}$. Let ex_0 be a best case execution (BCE) in which:

- processes p_0, l_0, l_2 and acceptors in Q_1 are correct;
- processes p_1, l_1 and acceptors in $\overline{Q_1}$ fail by crashing at the beginning of ex_0.

Such an execution is possible since the sets $\{p_0, l_0, l_2\} \cup Q_1$ and $\{p_1, l_1\} \cup \overline{Q_1}$ do not intersect.

In ex_0, correct proposer p_0 proposes 0 at the beginning of round 1, at time t_0 (i.e., p_0 sends $m0$). Since ex_0 is a BCE, the system is synchronous in the first two rounds of ex_0 (i.e., during $[t_0, t_0+2\Delta]$). Hence, all round 1 and 2 messages exchanged among all correct processes are delivered in ex_0. Since A is $(1, \boldsymbol{Q_1})$*–fast*, l_0 and l_2 learn 0 by the end of round 2 (i.e., in two message-delays).

$\underline{ex_1}$. Let ex_1 be the best-case execution (BCE) in which:

- processes p_1, l_1 and acceptors in Q_2 are correct;
- processes p_0, l_0, l_2 and acceptors in $\overline{Q_2}$ fail by crashing at the beginning of ex_1.

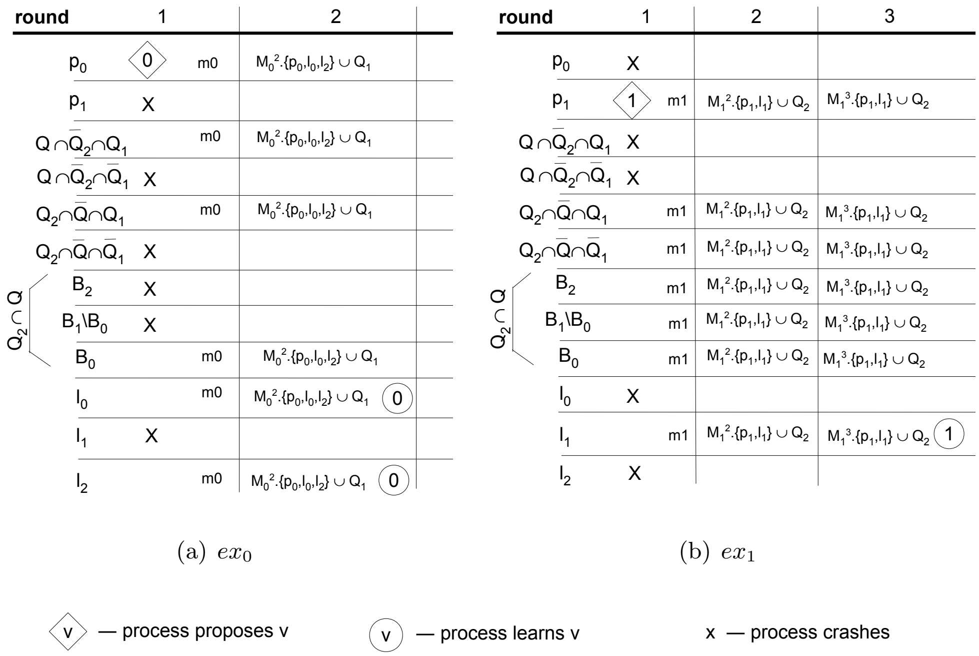
Such an execution is possible since the sets $\{p_1, l_1\} \cup Q_2$ and $\{p_0, l_0, l_2\} \cup \overline{Q_2}$ do not intersect.

In ex_1, correct proposer p_1 proposes 1 at the beginning of round 1, at time t_0 (i.e., p_1 sends $m1$). Since ex_1 is a BCE, the system is synchronous in the first three rounds of ex_1 (i.e., during $[t_0, t_0 + 3\Delta]$). Hence, all round 1-3 messages exchanged among all correct processes are delivered in ex_1. Since A is $(2, \boldsymbol{Q_2})$*–fast*, l_1 learns 1 by the end of round 3 (i.e., in three message-delays).

Executions ex_0 and ex_1 are depicted in Figure 16, where we show which messages are delivered by the end of a given round, as well as critical steps (like proposing or learning a value). We now construct 3 additional (partial) executions in which we reach a desired contradiction. These executions are depicted in Figure 17.

$\underline{ex_2 \text{ (Fig. 17(a))}}$. Let ex_2 be a partial execution in which:

- processes p_0, l_0 and acceptors in Q are correct;



(a) ex_0

(b) ex_1

(c) Legend

Fig. 16. Illustration of the partial executions used in the proof of Theorem 6 (executions ex_0 and ex_1). Only acceptors that belong to the set $Q_2 \cup Q$ are depicted.

– processes p_1, l_1, l_2 and acceptors in $\overline{Q}$ fail by crashing at the beginning of round 3 of ex_2, as detailed below.

Such an execution is possible since the sets $\{p_0, l_0\} \cup Q$ and $\{p_1, l_1, l_2\} \cup \overline{Q}$ do not intersect.

In ex_2, both proposers p_0 and p_1 propose values 0 and 1, respectively, at the beginning of round 1 (i.e., p_0 and p_1 send $m0$ and $m1$, respectively). Messages sent in the first two rounds of ex_2 are delivered as follows (see also Fig. 17(a)):

– (Round 1 messages.) By the end of round 1: processes in $\{p_0, l_0, l_2\} \cup \overline{Q_2}$ receive $m0$, while processes in $\{p_1, l_1\} \cup Q_2$ receive $m1$. Moreover, acceptors in B_2 receive the message from the correct proposer p_0 (i.e., message $m0$) in round 2, while those in B_1 receive $m0$ in round 3. No other process receives $m1$ (since p_1 crashes in ex_2).
– (Round 2 messages) The following round 2 messages are delivered by the end of round 2:
  - from $\{p_0, l_0, l_2\} \cup Q$ to p_0, l_0, l_2, $Q \cap \overline{Q_2}$ and B_2. In other words, processes in $\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2}) \cup B_2$ receive the set of messages $M_2^2.\{p_0, l_0, l_2\} \cup Q$ by the end of round 2. Notice that, at the end of round 1, processes in $Q \cap Q_2 = B_1 \cup B_2$ cannot distinguish ex_2 from ex_1 and play 1 in round 2 of ex_2. Hence, $M_2^2.\{p_0, l_0, l_2\} \cup Q$ is identical to the union of $M_2^2.\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2})$ and $M_1^2.Q \cap Q_2$.
  - from $\{p_1, l_1\} \cup Q_2$ to p_1, l_1, $Q_2 \cap \overline{Q}$ and B_1. In other words, processes in $\{p_1, l_1\} \cup (Q_2 \cap \overline{Q}) \cup B_1$ receive the set of messages $M_2^2.\{p_1, l_1\} \cup Q_2$ by the end of round 2. Notice that, at the end of round 1, processes in $\{p_1, l_1\} \cup Q_2$ cannot distinguish ex_2 from ex_1 and play 1 in round 2

of ex_2. Hence, these processes send identical messages in round 2 in both ex_2 and ex_1 and, therefore, $M_2^2.\{p_1, l_1\} \cup Q_2 = M_1^2.\{p_1, l_1\} \cup Q_2$.

Moreover, acceptors in B_1 receive the round 2 messages sent by processes in $\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2})$ (i.e., the set of messages $\mu = M_2^2.\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2})$) in round 3 (see Fig. 17).

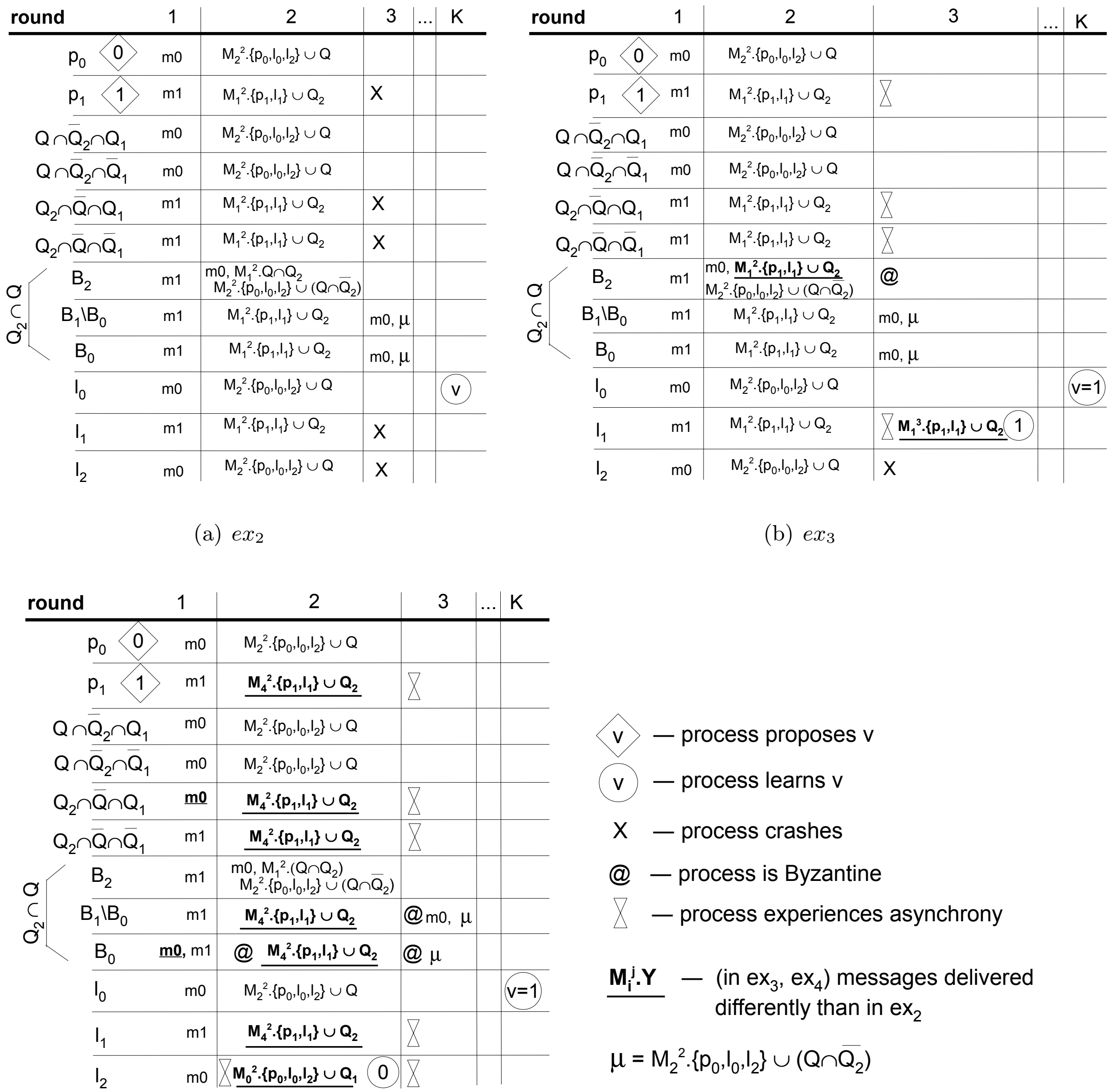
Finally, no other round 2 message is delivered in ex_2. (This is possible, since the only remaining round 2 messages are (a subset of) those sent by/to crash faulty processes in $\{p_1, l_1\} \cup \overline{Q}$). In particular, note that processes in $\{p_0, l_0, l_2\} \cup B_2 \cup (Q \cap \overline{Q_2})$ never receive any round 2 message sent by acceptors $Q_2 \cap \overline{Q}$.

Notice that, in ex_2, by the end of round 2, all processes in $\{p_0, p_1, l_0, l_1, l_2\} \cup Q_2 \cup Q$ receive all the messages sent in the first two rounds by (a) at least one proposer, (b) some quorum of acceptors, and (c) at least one learner. Processes in $\{p_1, l_1\} \cup (Q_2 \cap \overline{Q}) \cup B_1$ cannot distinguish, at the end of round 2, ex_2 from ex_1, while processes in $\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2}) \cup B_2$ cannot wait for any additional round 1 or round 2 message since these processes received all the messages sent by correct processes in the first two rounds. Therefore, in ex_2 no process in $\{p_0, p_1, l_0, l_1, l_2\} \cup Q_2 \cup Q$ waits for any additional message before moving to round 3.

At the beginning of round 3, processes in $\{p_1, l_1, l_2\} \cup \overline{Q}$ fail by crashing such that no process receives any message sent by some of these processes in round 3. Furthermore, assume that in every round $j \geq 3$, all round j messages exchanged among correct processes are delivered by the end of round j. Since (a) p_0 is correct in ex_2 and proposes a value, (b) there is a quorum of correct acceptors $Q \in \boldsymbol{Q^{(3)}}$, (c) the system is eventually synchronous, and (d) A implements $(\boldsymbol{Q^{(3)}}, \boldsymbol{B})$–*consensus*, eventually a correct learner l_0 learns some value $v \in \{0, 1\}$, say in round K, when partial execution ex_2 ends.

<u>ex_3 (Fig. 17(b))</u>. Let ex_3 be a partial execution identical to ex_2, except that, in ex_3:

1. Acceptors in B_2 receive, in round 2 of ex_3 (in addition to the messages they receive in ex_2), round 2 messages sent by processes in $\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$, i.e., $M_3^2.\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$. Recall here that processes in $\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$ play 1 in ex_2 (and hence in ex_3) and cannot distinguish at the end of round 1 ex_3 and ex_2 from ex_1; therefore, $M_3^2.\{p_1, l_1\} \cup (Q_2 \cap \overline{Q}) = M_1^2.\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$. Hence, acceptors in B_2 receive, by the end of round 2 of ex_3, all the messages in $M_1^2.\{p_1, l_1\} \cup Q_2$. Moreover, acceptors in B_2 are Byzantine in ex_3. They violate algorithm A in round 3 by sending the same message to l_1 as in the round 3 of ex_1 (i.e., as if acceptors in B_2 received *only* messages $M_1^2.\{p_1, l_1\} \cup Q_2$, in round 2). Otherwise, acceptors in B_2 send the same messages as in ex_2 to all other processes in rounds 3 to K (i.e., they "forget" they received messages $M_1^2.\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$ in round 2).
2. Processes in $\{p_1, l_1\} \cup \overline{Q}$ do not crash in ex_3 (the only process that crashes in ex_3 is the learner l_2). However, due to asynchrony, no message sent in round j, $j \leq K$ by some process in $\{p_1, l_1\} \cup \overline{Q}$ is delivered in ex_3, except: (a) round 1 and 2 messages as in ex_2 and (b) round 3 messages sent by processes in $\{p_1, l_1\} \cup (Q_2 \cap \overline{Q})$ to learner l_1. Other messages sent by processes in $\{p_1, l_1\} \cup \overline{Q}$ are in transit in ex_3.
3. In round 3 of ex_3, all round 3 messages sent by processes in $\{p_1, l_1\} \cup Q_2$ (including those sent by Byzantine acceptors in B_2) are delivered to l_1; other messages sent to l_1 are delayed and are in transit in ex_3. Since benign processes $\{p_1, l_1\} \cup Q_2$ do not distinguish round 2 of ex_3 from round 2 of ex_1, they send the same messages in round 3 of ex_3 as in round 3 of ex_1.



(a) ex_2

(b) ex_3

(c) ex_4

(d) Legend

Fig. 17. Illustration of the partial executions used in the proof of Theorem 6 (executions ex_2, ex_3 and ex_4). Only acceptors that belong to the set $Q_2 \cup Q$ are depicted. In executions ex_3 and ex_4, messages that are delivered differently than in ex_2 are emphasized. The set of messages $M_2^2.\{p_0, l_0, l_2\} \cup (Q \cap \overline{Q_2})$ is denoted by μ.

Hence, at the end of round 3, l_1 cannot distinguish ex_3 from ex_1, receives the set of messages $M_1^3.\{p_1, l_1\} \cup Q_2$ and learns 1.

Other round 3 and later messages in ex_3 are delivered as in ex_2. Hence, a correct learner l_0 (the only faulty processes in ex_3 are those in $\{l_2\} \cup B_2$ to which l_0 does not belong) cannot distinguish ex_3 and ex_2. Therefore, l_0 learns a value v by the end of round K when partial execution ex_3 ends. Since both l_1 and l_0 are correct in ex_3, by the *Agreement* property, v must equal 1.

<u>ex_4 (Fig. 17(c))</u>. Let ex_4 be a partial execution in which:

- all processes are correct, except acceptors in B_1;
- acceptors in B_1 are Byzantine, as detailed below. Recall that $B_0 \subseteq B_1$.

At the beginning of ex_4, p_0 proposes 0, while p_1 proposes 1. In round 1 of ex_4 the message are delivered exactly as in round 1 of ex_2, except that (see also Fig.17(c)):

1. benign acceptors in Q_1 (including those in $Q_2 \cap \overline{Q} \cap Q_1$) receive $m0$, but not $m1$, and
2. acceptors in B_0 receive both $m0$ and $m1$.

In round 2 and later rounds, B_0 plays 1 to processes other than l_2. Moreover, in round 2, all acceptors in Q_1 (including those in B_0), as well as processes in $\{p_0, l_0, l_2\}$, play 0 to l_2. Here, benign processes in $\{p_0, l_0, l_2\} \cup Q_1$ obey the algorithm — they cannot distinguish round 1 of ex_4 from that of ex_0. Moreover, in round 2 of ex_4, l_2 receives all the round 2 messages from processes in $\{p_0, l_0, l_2\} \cup Q_1$ and l_2 receives only those messages — all other messages sent to l_2 are in transit in ex_4. Clearly, at the end of round 2, l_2 cannot distinguish ex_4 from ex_0 — in round 2, l_2 receives the set of messages $M_0^2.\{p_0, l_0, l_2\} \cup Q_1$ as in ex_0. Hence, l_2 learns 0 by the end of round 2 in ex_4. Finally, all messages sent by l_2 in round 3 and later are delayed, and are in transit in ex_4.

All other round 2 messages are delivered following the pattern of round 2 of ex_2. Hence, just like in ex_2, in ex_4, no process in $X = \{p_0, l_0\} \cup (Q \cap \overline{Q_2}) \cup B_2$ receives any round 2 message from any acceptor in $Q_2 \cap \overline{Q}$. Therefore, processes belonging to set X miss the information that processes in $Q_2 \cap \overline{Q} \cap Q_1$ received $m0$ in the first round. Moreover, since Byzantine acceptors in B_0 play 1 to all processes but l_2, processes in X cannot distinguish ex_4 from ex_2 at the end of round 2.

Starting from round 3, Byzantine acceptors in B_1 forge their state as if they received the round 2 messages as in ex_2 and ex_1, i.e., as if all the processes in $\{p_1, l_1\} \cup Q_2$ played 1 to acceptors in B_1 in round 2 of ex_4 (which is actually the case, except for the processes in $Q_2 \cap \overline{Q} \cap Q_1$), and acceptors in B_1 received these messages. This is possible since the messages sent in the round 2 of the best-case execution ex_1 are not authenticated.

Finally, messages sent starting from round 3 by/to processes in $\{p_1, l_1\} \cup \overline{Q}$ are delayed and are in transit in ex_4. Note that the set $\{p_1, l_1\} \cup \overline{Q}$ includes all processes other than $\{l_2\} \cup B_1$ (which are either also delayed or Byzantine) that can distinguish ex_4 from ex_2. All remaining messages in round 3 and later are delivered as in ex_2. This impedes correct learner l_0 from distinguishing ex_4 and ex_2. Hence, l_0 learns a value v by the end of round K, when ex_4 ends. Since v equals 1 (see ex_3), and both l_0 and l_2 are correct, in ex_5 *agreement* is violated.

Just like in the proof of Theorem 3, the assumption that Property 3 of RQS does not hold is critical in reaching a violation of *agreement* using the above sequence of executions ex_0 to ex_4. Namely, if Property 3 holds, then $P_{3a}(Q_2, Q, B'_1)$ holds (implying $B_2 = Q_2 \cap Q \setminus B'_1 \notin \boldsymbol{B}$), in which case we cannot have ex_3, or $P_{3b}(Q_2, Q, B'_1)$ holds (implying $B_0 \setminus B'_1 \neq \emptyset$, i.e., $B_0 \nsubseteq B'_1$, where $B_0 = Q_1 \cap Q_2 \cap Q$), in which case we cannot have ex_4. □

5 Related Work

In this section, we compare the techniques used in our atomic storage and consensus algorithms to existing algorithms; thus, we complement the comparison of our paper and RQS to previous work that we gave in Section 2.2.

Our atomic storage algorithm (Section 3.2) is inspired by techniques used in the regular [33] finite write (FW) terminating[9] storage algorithm of [2] and the wait-free atomic storage algorithm of [20]. Both of these algorithms are optimally resilient, like our algorithm, in terms of tolerating Byzantine servers (for tolerating Byzantine clients see, e.g., [3, 18, 23]). Unlike these (or other existing) algorithms, our atomic wait-free storage algorithm is the first to combine optimal resilience and best-case optimal latency in the non-threshold failure model.

However, our storage algorithm (deliberately) features unbounded worst case message-complexity and uses unbounded storage at servers (i.e., a server stores an entire history of a shared variable). Hence, it may seem that RQS have an undesirable side-effect of ruining other complexity metrics while optimizing the best-case time-complexity. However, achieving atomic semantics (and even a weaker, regular one), in a wait-free manner while precluding servers from storing the entire history, is unfeasible without using some non-trivial signaling scheme between the readers and the writer [2, 9]. Such schemes include, for example, the "Listeners" pattern of [42] (in which, roughly, concurrent readers subscribe at servers for updates on concurrent writes and, as such, is not applicable to our storage model), the "freezing" technique of [20] (applicable to our model), bounded implementation techniques of [3], as well as techniques used in [5]. Similarly, optimizing Byzantine fault-tolerant storage (worst-case) complexity is not trivial [3, 21]. We opt not to address these issues in this paper since we believe that doing so would not contribute to better understanding of RQS. Our algorithms serve to illustrate how RQS can be used to build algorithms with optimal best case time complexity — they do not aim to optimize other complexity metrics. Integrating the above techniques here would not contribute to deeper understanding of RQS and might obfuscate the point of this paper.

As an illustration of why using RQS does not require servers to store an entire history of a shared variable, nor imposes unbounded message complexity, recall the simple variation of [4], that we described in our example in Section 1.2). The mentioned algorithm implements optimally (crash) resilient wait-free atomic storage. Moreover, the algorithm makes use of RQS; in the particular case of a system consisting of 5 servers, class 1 quorums are all subsets of 4 or more servers, whereas subsets containing 3 servers are class 2 quorums. Finally, all servers store 2 copies of the shared variable, and reads and writes complete in at most 2 rounds.

Our consensus algorithm (Section 4.2) is itself inspired by the PBFT algorithm [7], from which it borrows the idea of view-change mechanisms (with significant differences in choosing the proposal value during a view-change). Namely, like PBFT, our algorithm proceeds in sequence of "views", which can be mapped to "ballots" in the Paxos algorithm [38]. Unlike PBFT, our algorithm does not implement state-machine replication, but rather a single-shot consensus instance.

Finally, proofs of our storage and consensus algorithms (Appendices A and B), rely extensively on the notion of a *basic subset* (intuited in Section 3.2) which simply denotes a set of servers not belonging to adversary $\boldsymbol{B}$. Basic subsets are similar to *cores* defined by Junqueira and Marzullo [28, 29]. They define cores (in broader context of a framework for tolerating dependent failures) as minimal sets of processes such that at least one process is correct in every execution. In contrast,

[9] In FW-terminating implementations, reads might not terminate in case there is an unbounded number of writes.

a basic subset is *any* set of processes such that at least one process is *benign* (i.e., correct or crash-faulty) in every execution. Hence, all cores are basic subsets, yet not all basic subsets are cores.

6 Concluding Remarks

This paper introduces the notion of *refined quorum systems* (RQS) and argues that this is a useful notion to reason about optimally resilient and efficient distributed object implementations assuming general adversary structures. We show that refined quorum systems are necessary and sufficient (or, in a sense, minimal) for implementing an important class of *atomic* objects, namely atomic storage and consensus. This minimality holds when we indeed require atomicity and do not rely on authentication primitives to cope with Byzantine failures in best-case executions.

Roughly speaking, denoting the best possible latency of an object implementation by l_1[10] (i.e., 1 round in the case of storage, or 2 message delays in the case of (Byzantine and asynchronous [35]) consensus, and by l_2 and l_3, incrementally, the next best possible latencies according to the corresponding metric, we proposed two RQS-based object implementations that achieve a latency of l_i whenever a quorum of class i is available and best-case conditions (namely, synchrony and no-contention) are met. Since Property 1 of RQS (defined on class 3 quorums) is anyway necessary for any resilient implementation of distributed storage and consensus in an asynchronous environment, there is no need for refining quorums further.

It might be important to notice here that the very notion of a refined quorum system helps highlight the information structure of optimally resilient and best-case efficient atomic object implementations (at least those implementing the abstractions of atomic storage or consensus). Basically, these implementations go through at most three "rounds" in best-case conditions and fall into a backup subprotocol in case of asynchrony or contention. A novel algorithmic scheme we used in both algorithms consists of appending the ids of (class 2) quorums, to written/proposed values. This is key to combining graceful degradation (i.e., achieving both latencies l_1 and l_2) with optimal resilience.

Our study opens several research directions. For example, it is intriguing to determine:

- the load and availability of RQS [44],
- how RQS can be optimally placed in the network [17],
- the extension of RQS with respect to asymmetric read and write quorums [43],
- how many RQS can be found given some adversary structure,
- how to devise algorithms that cope with unknown RQS/adversary structures, and
- how RQS can be expressed in frameworks for tolerating non-independent and identically distributed (non-IID) failures, other than general adversary structures (in particular, in the core/survivor framework of [29]).

Moreover, it would be interesting to carefully look into non-atomic semantics, e.g., regular or safe storage [33]. Recent results (in the threshold-based context) suggest that some (yet not all) properties of our RQS are necessary and sufficient even for achieving optimal best-case complexity of weaker object implementations. Namely [2, 21] suggest that Properties 1 and 3a of RQS are necessary and sufficient for characterizing non-atomic best-case efficient storage implementations. These properties correspond to the special case of RQS where $\boldsymbol{QC_1} = \emptyset$. Finally, it would also

[10] This can be measured by the best possible latency in synchronous, uncontended and failure-free situations.

be interesting to look into atomic object implementations that use authentication in best-case executions. The lower bounds of [35], stated in the threshold-based context, suggest that Properties 1 and 2 are necessary and sufficient for characterizing best-case efficient and optimally resilient consensus implementations regardless of whether authentication is used in the best-case. These properties correspond to the special case of RQS where $\boldsymbol{QC_2} = \boldsymbol{QC_1}$.

Finally, we note that the preliminary, conference version of this paper [22], was erroneous, notably in the way it stated Property 3 of RQS (please see Appendix C for more details).

Acknowledgments

We are very grateful to anonymous reviewers for pointing out a mistake in the conference version of this paper. Their very detailed comments significantly improved the paper. We also thank Hagit Attiya for her detailed comments on earlier drafts of this paper. Finally, we thank Christian Cachin, Christof Fetzer, Petr Kouznetsov, Ron R. Levy, Dahlia Malkhi and Eric Ruppert for their helpful comments and useful feedback.

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A Correctness of the atomic storage algorithm

In this Appendix, we prove the correctness of our atomic storage algorithm of Section 3.2. For simplicity of presentation, we introduce the following notation and definitions:

- We say that the writer *attaches* timestamp ts to value val, if the writer invokes $\mathsf{write}(val)$ and ts equals the writer's local variable ts after the writer executes line 1, Fig. 5 in $\mathsf{write}(val)$;
- We say that a (timestamp/value) pair $c = \langle ts, val \rangle$ is *valid*, if the writer attached ts to val, or if $c = \langle 0, \perp \rangle$ (otherwise, c is called *invalid*).
- If some reader r executes line 35, Fig. 7, and assigns some pair $c = \langle ts, val \rangle$ to c_{sel}, we say r *selects* pair c;
- We say that a server *responds to*, or *acks* a $\mathsf{wr}\langle ts, *, *, rnd \rangle$ (resp., $\mathsf{rd}\langle tsr, rnd \rangle$) message from client, if a server sends a $\mathsf{wr\_ack}\langle ts, rnd \rangle$ (resp., $\mathsf{rd\_ack}\langle tsr, rnd, * \rangle$) to the client.
- We say that a benign server s_i *stores* pair $c = \langle c.ts, c.val \rangle$ (in *slot* $rnd \in \{1, 2, 3\}$), if, at some point in time, $history_i[c.ts, rnd].pair = c$. If, additionally, $Q \in history_i[c.ts, rnd].sets$ for some quorum Q, we say that s_i stores c (in slot rnd) *with* quorum Q;
- We denote by B_{ex} the set that contains all Byzantine servers in some execution ex (we assume $B_{ex} \in \boldsymbol{B}$).

Definition 5. *Consider a set of elements S and an adversary for S, $\boldsymbol{B}$. We say that $Q \subseteq S$ is a* basic *(resp.,* large*) subset (of S), if Q is not a subset of any element (resp., a union of any two elements) of an adversary structure, i.e., $Q \notin \boldsymbol{B}$ (resp., $\forall B_1, B_2 \in \boldsymbol{B}$: $Q \nsubseteq (B_1 \cup B_2)$).*

We first prove atomicity and then we proceed to wait-freedom and complexity. To prove atomicity, we first prove few simple lemmas.

Lemma 1. ***Size of basic sets.*** *In every execution of our storage algorithm, any basic subset of servers contains at least one benign server.*

Proof. The lemma follows directly from the definition of a basic subset (Definition 5). □

Lemma 2. ***Size of large sets.*** *In any execution ex of our storage algorithm, for any large subset T_2 of servers, there is a basic subset $T_1 \subseteq T_2$ that contains only benign servers.*

Proof. By Definition 5, for any large subset T_2, $T_2 \setminus B_{ex}$ is a basic subset. By definition of B_{ex}, $T_2 \setminus B_{ex}$ contains only benign servers in ex. Hence the lemma. □

The following two lemmas follow directly from our assumption of benign readers and by trivial inspection of the read pseudocode in Fig. 7.

Lemma 3. ***Returned values.*** *If read rd by reader r returns value val, then r selected pair $c = \langle ts, val \rangle$, for some timestamp ts.*

Lemma 4. ***Values written by readers.*** *If some reader r sends a $\mathsf{wr}\langle ts, v, *, * \rangle$ to servers, then r selected $\langle ts, v \rangle$.*

Lemma 5. ***Validity of selected pairs.*** *If reader r selects pair $c = \langle c.ts, c.val \rangle$ in some read, then c is valid.*

Proof. Suppose by contradiction that some read selects pair c, such that c is invalid. Let rd be the first read (according to the global clock) to select an invalid pair c at time t. Therefore, up to time t, by Lemma 4, readers send only wr messages containing valid pairs and benign servers store only valid pairs.

Since r selects c, predicate $safe(c)$ holds in rd, i.e., servers from a *basic* subset T have sent a rd_ack message containing c in their $history[c.ts, 1]$ or $history[c.ts, 2]$ variables. By Lemma 1 at least one benign server $s_i \in T$ stored an invalid pair c before time t. A contradiction. □

Lemma 6. ***Validity of stored pairs.*** *Benign servers store only valid pairs.*

Proof. Follows from Lemmas 4 and 5. □

Lemma 7. ***No ambiguity.*** *No two benign servers ever store different pairs with the same timestamp.*

Proof. By Lemma 6, the assumption that the writer is benign and the fact that the writer never attaches different timestamps to the same value. □

Lemma 8. ***Sticky values.*** *For any $rnd \in \{1, 2, 3\}$ and benign server s_i, once $history_i[ts, rnd].pair \neq \langle 0, \perp \rangle$ it is never modified.*

Proof. Immediate from the condition in line 4 of the server pseudocode (Fig. 6). □

Lemma 9. ***Sticky sets.*** *For any $rnd \in \{1, 2, 3\}$ and benign server s_i, once s_i adds the id of some quorum Q to $history_i[ts, rnd].sets$, s_i never removes Q from $history_i[ts, rnd].sets$.*

Proof. Trivially, by inspection of the server code (Fig. 6). □

Before proceeding with the proof, we define the following three global predicates, extensively used in the remainder of the proof.

1. $V_1(c, Q)$ holds if and only if there is a basic subset $T_Q \subseteq Q$ that contains *only* benign servers, such that every server $s_i \in T_Q$ stores c in slot 1 (i.e., $\forall s_i \in T_Q : history_i[c.ts, 1].pair = c$)

2. $V_2(c, Q)$ holds if and only if there is a benign server $s_i \in Q$ that stores c in slot 2 (i.e., $history_i[c.ts, 2].pair = c$);

3. $V_3(c, Q)$ holds if and only if there is a class 2 quorum Q_2 and a set B that belongs to the adversary structure $\boldsymbol{B}$, such that $P_{3b}(Q_2, Q, B)$ holds and every server s_i from $Q_2 \cap Q \setminus B$ is benign and s_i stores c in slot 1 with quorum Q_2 (i.e., $\forall s_i \in Q_2 \cap Q \setminus B : history_i[c.ts, 1] = \langle c, \boldsymbol{Set_i} \rangle \wedge Q_2 \in \boldsymbol{Set_i}$).

The following lemma relates the global predicate $V_j(c, Q)$ to read predicate $valid_j(c, Q)$ (for $j \in \{1, 2, 3\}$).

Lemma 10. *Assume $V_j(c, Q)$ holds at time t in some execution ex. Then in* read *rd invoked after t, $Q \in$ **Responded** implies $valid_j(c, Q)$, for any $j \in \{1, 2, 3\}$.*

Proof. By Lemmas 8 and 9 once the predicate $V_j(c, Q)$ (for $j \in \{1, 2, 3\}$) becomes true in some execution ex, it remains always true in ex. Then, since $Q \in$ ***Responded***, all benign servers from Q responded to at least one rd message in rd (lines 52-53, Fig. 7). Finally, since rd is invoked in ex after $V_j(c, Q)$ becomes true in ex, it is straightforward to see that in rd $valid_j(c, Q)$ holds (for any $j \in \{1, 2, 3\}$). □

The following lemma is crucial in proving atomicity.

Lemma 11. ***Locking the value.*** *If for every quorum Q, at least one of properties $V_j(c, Q)$ holds by time t (for some $j \in \{1, 2, 3\}$), for some pair c, then reader r cannot select pair c' in a complete* read *rd invoked after t, such that $c'.ts < c.ts$.*

Proof. Let t_1 be the time when the first round of rd completes; by lines 26 and 52-53 of Fig. 7, for $t \geq t_1$, ***Responded*** contains at least one quorum Q_r. By reader pseudocode, reader r cannot select any pair in rd before t_1. Fix $j \in \{1, 2, 3\}$, such that $V_j(c, Q_r)$ holds before rd is invoked.

Assume by contradiction that r can select c' in rd. Then, $highCand(c')$ holds (line 9, Fig. 7) in rd after t_1. Since $highCand(c')$, at least one of the following must hold: (a) after t_1, there is no server s_i, such that $read(c, i)$ holds in rd, or (b) $invalid(c)$ holds in rd after t_1. However, since $Q_r \in$ ***Responded*** (by time t_1) and since $V_j(c, Q_r)$ holds before rd is invoked, by Lemma 10, $valid_j(c, Q_r)$ holds in rd after t_1. Moreover, by definition of $highest\_ts$ (line 29, Fig. 7), in rd, $highest\_ts \geq c.ts$. Therefore, $invalid(c)$ cannot hold in rd after t_1. Moreover, $valid_j(c, Q_r)$ trivially implies $read(c, i)$ for some $s_i \in Q_r$. A contradiction. □

Now we prove atomicity of read operations with respect to write operations.

Lemma 12. read/write ***atomicity.*** *If* read *rd is complete and it follows some complete $wr =$* write*(v), then rd does not return a value older than v.*

Proof. Having Lemma 3 in mind, it is sufficient to show that the reader in rd does not select c, such that $c.ts < ts$, where ts is the timestamp attached by the writer to v.

First, suppose that wr completes in a single round. Then, all benign servers of some class 1 quorum Q_1 store pair $\langle ts, v \rangle$ in slot 1. By Property 2 of RQS, and Definition 5, for every quorum Q, $Q_1 \cap Q$ is a large subset, and every large subset is a superset of a basic subset that contains only benign servers (by Lemma 2). Therefore, for every quorum Q there is a basic subset T_Q that contains only benign servers such that $T_Q \subseteq Q$ and $T_Q \subseteq Q_1$. Hence, for every quorum Q property $V_1(\langle ts, v \rangle, Q)$ holds by the end of wr. Finally, by Lemma 11, rd does not select c, such that $c.ts < ts$.

Now, suppose that wr completes in two or three rounds. Then, all benign servers of some quorum Q' store pair $\langle ts, v \rangle$ in slot 2. Since any quorum intersects with any other quorum Q in a basic subset (by Property 1 of RQS, and Definition 5), $Q' \cap Q$ is a basic subset that contains at least one benign server (by Lemma 1). Therefore, for every quorum Q there is a benign server s_{i_Q} such that $s_{i_Q} \in Q$ and $s_{i_Q} \in Q'$. Hence, for every quorum Q property $V_2(\langle ts, v \rangle, Q)$ holds by the end of wr. Finally, by Lemma 11, rd does not select c, such that $c.ts < ts$.

Now we proceed to proving atomicity of read operations. First, we prove one auxiliary lemma.

Lemma 13. ***Previous slots.*** *If a benign server stores c in slot 2 with class 2 quorum Q_2, then all benign servers from Q_2 already stored c in slot 1. Similarly, if a benign server stores c in slot 3, then all benign servers from some quorum already stored c in slot 2.*

Proof. To prove the first part of the lemma, notice that, by code inspection, (a) benign servers can store c in slot 2 with Q_2 only upon receiving a $\mathsf{wr}\langle c.ts, c.val,\ \boldsymbol{Set}, 2\rangle$ message with $Q_2 \in \boldsymbol{Set}$, and (b) only the writer can send a $\mathsf{wr}\langle *, *,\ \boldsymbol{Set}, 2\rangle$ message with non-empty $\boldsymbol{Set}$. Moreover, the writer sends such a message with $Q_2 \in \boldsymbol{Set}$, only in the second round of $\mathsf{write}(c.val)$ after the writer receives responses from all servers from Q_2 in round 1 of $\mathsf{write}(c.val)$. Hence, by the end of round 1 of $\mathsf{write}(c.val)$, all benign servers from Q_2 store c in slot 1.

The proof of the second part of the lemma closely follows that of the first part: (a) benign servers can store c in slot 3 with Q_2 only upon receiving a $\mathsf{wr}\langle c.ts, c.val, *, 3\rangle$ message, and (b) only the writer can send such a message and only in the third round of $\mathsf{write}(c.val)$ after the writer receives responses from all servers from some quorum in round 2 of $\mathsf{write}(c.val)$. Hence, by the end of round 2 of $\mathsf{write}(c.val)$, all benign servers from some quorum store c in slot 2. $\square$

Lemma 14. read *atomicity.* *If* read *rd is complete and it follows some complete* read *rd' that returns v', then rd does not return a value older than v'.*

Proof. Having Lemma 3 in mind, it is sufficient to show that the reader in rd does not select c, such that $c.ts < ts'$, where $c' = \langle ts', v'\rangle$ was selected in rd'. Moreover, by Lemma 11, it is sufficient to show that, in any execution ex and for every quorum Q, at least one of the properties $V_j(c', Q)$ (for $j \in \{1, 2, 3\}$) holds by the time rd' completes.

We consider three exhaustive cases, where : (1) rd' completes in a single round, (2) rd' completes in 2 rounds, and (3) rd' completes in at least 3 rounds.

1. If rd' completes in a single round, then, at the end of the first round of rd', $BCD(c', 1, rnd)$ holds, for some $rnd \in \{1, 2, 3\}$. We consider the following three exhaustive cases where: (a) $BCD(c', 1, 1)$ holds, (b) $BCD(c', 1, 2)$ holds, and (c) $BCD(c', 1, 3)$ holds.
   (a) In case $BCD(c', 1, 1)$ holds in rd', there are class 1 quorums Q_1 and Q'_1 (possibly $Q_1 = Q'_1$), such that, by the end of round 1 of rd', for all benign servers $s_i \in Q_1 \cap Q'_1 = X$, $history[i, ts', 1].pair = c'$ in rd'. Hence, by the end of round 1 of rd', all benign servers $s_i \in X$ stored c' in slot 1. By Property 2 of RQS and Definition 5, an intersection of a pair of class 1 quorums with any quorum Q is a large subset. Hence, $X \cap Q$ is a large subset, and, by Lemma 2, every large subset is a superset of a basic subset that contains only benign servers. Therefore, for every quorum Q there is a basic subset T_Q that contains only benign servers such that $T_Q \subseteq Q$ and $T_Q \subseteq X$. Hence, for every quorum Q property $V_1(c', Q)$ holds by the time rd' completes.
   (b) In case $BCD(c', 1, 2)$ holds in rd', there is class 1 quorum Q_1 and class 2 quorum Q_2, such that, by the end of round 1 of rd', for all benign servers $s_i \in Q_1 \cap Q_2 = X$, $history[i, ts', 2] = \langle c', \boldsymbol{QC''_2}\rangle$ and $Q_2 \in \boldsymbol{QC''_2}$ in rd'. Hence, by the end of round 1 of rd', all benign servers $s_i \in X$ stored c' in slot 2 with Q_2. Moreover, by Lemma 13, all benign servers from Q_2 stored c' in slot 1. By Property 3 of RQS, in any execution ex, for every quorum Q at least one of the properties $P_{3a}(Q_2, Q, B_{ex})$ and $P_{3b}(Q_2, Q, B_{ex})$ holds. We now show that if $P_{3a}(Q_2, Q, B_{ex})$ (resp., $P_{3b}(Q_2, Q, B_{ex})$) holds, then property $V_1(c', Q)$ (resp., $V_2(c', Q)$) holds in ex by the time rd' completes:
      i. In case $P_{3a}(Q_2, Q, B_{ex})$ holds, $T_Q = Q_2 \cap Q \setminus B_{ex}$ is a basic subset, that contains only benign servers (by Property 3a of RQS and definition of B_{ex}). Hence, all (benign) servers in T_Q stored c' in slot 1, before rd' completes. Since $T_Q \subseteq Q$, $V_1(c', Q)$ holds in ex by the time rd' completes.

ii. In case $P_{3b}(Q_2, Q, B_{ex})$ holds, $X \cap Q = Q_1 \cap Q_2 \cap Q \nsubseteq B_{ex}$, i.e., $X \cap Q$ contains at least one benign server s_i. Since $s_i \in X$, s_i stores c' in slot 2 before rd' completes. Moreover, since $s_i \in Q$, $V_2(c', Q)$ holds in ex by the time rd' completes.

(c) In case $BCD(c', 1, 3)$ holds in rd', there is class 1 quorum Q_1 and a quorum Q', such that, at the end of round 1 of rd', for all benign servers $s_i \in Q_1 \cap Q = X$, $history[i, ts', 3].pair = c'$. Since, by Property 1 of RQS, any quorum intersection is a basic subset, by Lemma 1, X contains at least one benign server s_i. Hence, by the end of the round 1 of rd' benign server $s_i$store c' in slot 3. Moreover, by Lemma 13, there is a quorum Q'' such that all benign servers from Q'' stored c' in slot 2 by the end of round 1 of rd'.

By Property 1 of RQS and Definition 5, $Q'' \cap Q'$ is a basic subset for every quorum Q. Moreover, every basic subset contains at least one benign server (by Lemma 1). Therefore, for every quorum Q there is a benign server $s_Q \in Q$ that stored c' in slot 2 before rd' completed. Hence, for every quorum Q property $V_2(c', Q)$ holds by the end of rd'.

2. If rd' completes in exactly two rounds, then $\exists rnd \in \{1, 2, 3\} : \boldsymbol{BCD}(c', 2, rnd) \neq \emptyset$ (line 41, Fig. 7). We consider the following two exhaustive cases, where, in rd': (a) $\boldsymbol{BCD}(c', 2, rnd) \neq \emptyset$ holds for $rnd \in \{2, 3\}$, and (b) $\boldsymbol{BCD}(c', 2, rnd) \neq \emptyset$ holds only for $rnd = 1$.

(a) In this case, a client that invoked rd' received acks for its $\mathsf{wr}\langle ts', v', \emptyset, 2\rangle$ message from at least a quorum Q' of servers (when executing the *writeback* procedure in line 42). Hence, by the time rd' completes, all benign servers from Q' store c' in slot 2.

By Property 1 of RQS and Definition 5, for every quorum Q, $Q' \cap Q$ is a basic subset. Moreover, every basic subset contains at least one benign server (by Lemma 1). Therefore, for every quorum Q there is a benign server $s_Q \in Q$ that stored c' in slot 2 before rd' completed. Hence, for every quorum Q property $V_2(c', Q)$ holds by the end of rd'.

(b) In this case, since $\boldsymbol{BCD}(c', 2, 1) \neq \emptyset$, Q_2 is an element of $\boldsymbol{BCD}(c', 2, 1)$ if: (i) Q_2 is a class 2 quorum that responded in the first round of rd', and (ii) there is a class 1 quorum Q_1, such that, for all servers from $X = Q_1 \cap Q_2$, $history[*, ts', 1].pair = c'$.

In the round 2 of rd' the reader sends the $\mathsf{wr}\langle ts', v', \boldsymbol{BCD}(c', 2, 1), 1\rangle$ to all servers. Since rd' completes in exactly two rounds, all servers from some (class 2) quorum from $\boldsymbol{BCD}(c', 2, 1)$ respond in the round 2 of rd' as well. Denote this quorum by Q_2'. Then, all benign servers from Q_2' store c' in slot 1 with Q_2' by the time rd' completes.

Since Q_2' is a class 2 quorum, by Property 3 of RQS, for every quorum Q, at least one of the following two properties holds in any execution ex: (i) $P_{3a}(Q_2', Q, B_{ex})$, or (ii) $P_{3b}(Q_2', Q, B_{ex})$. We now show that if $P_{3a}(Q_2', Q, B_{ex})$ (resp., $P_{3b}(Q_2', Q, B_{ex})$) holds, then property $V_1(c', Q)$ (resp., $V_3(c', Q)$) holds in ex by the time rd' completes:

i. In case $P_{3a}(Q_2', Q, B_{ex})$ holds, $T_Q = Q_2' \cap Q \setminus B_{ex}$ is a basic subset that contains only benign servers (by Property 3a of RQS and definition of B_{ex}). Since $T_Q \subseteq Q_2'$, all (benign) servers from T_Q stored c' in slot 1, before rd' completes. Since $T_Q \subseteq Q$, $V_1(c', Q)$ holds in ex by the time rd' completes.

ii. In case $P_{3b}(Q_2', Q, B_{ex})$ holds, $Q_2' \cap Q \setminus B_{ex}$ is a set that contains only benign objects s_i, such that every s_i stored c' in slot 1 with Q_2' before rd' completes. Hence, $V_3(c', Q)$ holds in ex by the time rd' completes.

3. If rd' completes in more than two rounds, then a client that invoked rd' received acks for its $\mathsf{wr}\langle ts', v', \emptyset, 2\rangle$ message from at least a quorum Q' of servers (when executing the *writeback* procedure in line 47, or line 49). Hence, by the time rd' completes, all benign servers from Q' store c' in slot 2. Applying Property 1 of RQS, it is not difficult to see that for every quorum

Q, $Q' \cap Q$ contains at least one benign server. Hence, $V_1(c', Q)$ holds for every quorum Q by the time rd' completes.

Theorem 7. ***Atomicity.*** *The algorithm in Figures 5, 6 and 7 is atomic.*

Proof. By Lemmas 5, 12 and 14. □

We proceed to prove the wait-freedom property. In the remainder of the proof we denote by Q_c a quorum that contains only correct servers.

First we prove two important auxiliary lemmas that provide an intuition behind the liveness of the algorithm provided a quorum that contains only correct servers, Q_c. Roughly speaking, the two lemmas state that it is not possible that, for some timestamp value pair c, $valid_2(c, Q_c)$ (resp., $valid_3(c, Q_c)$) holds yet that $safe(c)$ does not hold (notice that $valid_1(c, Q)$ trivially implies $safe(c)$, for any quorum Q).

Lemma 15. ***Liveness of*** $valid_2$ ***predicate.*** *No server $s_i \in Q_c$ stores c in slot 2, before there is a basic subset $T \subseteq Q_c$ such that all $s_j \in T$ stored c in slot 1.*

Proof. By the algorithm's pseudocode, (a) correct server s_i stores a pair c in slot 2 only after some client sends $\mathsf{wr}\langle c.ts, c.val, *, 2\rangle$ to s_i. Moreover, before any client $clnt$ sends $\mathsf{wr}\langle c.ts, c.val, *, 2\rangle$ to servers, $clnt$ has already received acks for its $\mathsf{wr}\langle c.ts, c.val, *, 1\rangle$ message from some quorum Q, except in case of a writeback in line 42 of Fig. 7 (where $clnt$ is a reader). In all other cases, $Q \cap Q_c$ is a basic subset (by Property 1 of RQS and Definition 5) — hence the lemma.

In the case of a writeback in line 42 of Fig. 7, assume, by contradiction, that there is a read rd that issues a message $\mathsf{wr}\langle c.ts, c.val, \emptyset, 2\rangle$, such that there is no basic subset $T \subseteq Q_c$, such that all servers from T previously stored c in slot 1. Moreover, let rd be the first such read according to the global clock that executes the *writeback* procedure in line 42, Fig. 7, at time t.

In this case, $\boldsymbol{BCD}(c, 2, 2)$ or $\boldsymbol{BCD}(c, 2, 3)$ are not empty in line 42 of rd. If $\boldsymbol{BCD}(c, 2, 2)$ (resp., $\boldsymbol{BCD}(c, 2, 3)$) are not empty, then there exist two class 2 quorum Q_2 and Q'_2 (resp., a class 2 quorum Q_2 and a quorum Q) such that, all benign servers from $Q_2 \cap Q'_2 = X$ (resp., $Q_2 \cap Q = X$), stored c in slot 2 (resp., slot 3). By Property 1 of RQS, Definition 5 and Lemma 1, there is at least one benign server in X, i.e., by the end of round 1 of rd at least one benign server received $\mathsf{wr}\langle c.ts, c.val, *, 2\rangle$ (resp., $\mathsf{wr}\langle c.ts, c.val, *, 3\rangle$) from some client $clnt'$. By algorithm pseudocode and by our assumption on rd, before time t, client $clnt'$ sends a message $\mathsf{wr}\langle c.ts, c.val, *, 2\rangle$ (resp., $\mathsf{wr}\langle c.ts, c.val, *, 3\rangle$) only upon $clnt'$ received acks for its $\mathsf{wr}\langle c.ts, c.val, *, 1\rangle$ message from some quorum Q' of servers (i.e., not in line 42 of Fig. 7). Note that $Q' \cap Q_c = T_c$ is a desired basic subset. A contradiction.

Lemma 16. ***Liveness of*** $valid_3$ ***predicate.*** *Let Q_2 be any class 2 quorum and B any element of adversary structure $\boldsymbol{B}$, such that $P_{3b}(Q_2, Q_c, B)$ holds. If every server $s_i \in Q_2 \cap Q_c \setminus B$ stored c in slot 1 with Q_2 (by time t), then (not later than t) there is a basic subset T such that $T \subseteq Q_c$, and all $s_j \in T$ stored c in slot 1.*

Proof. Denote the set $Q_2 \cap Q_c \setminus B$ by X. Obviously, if $X \notin \boldsymbol{B}$, X is the desired basic subset and the lemma follows. Therefore, in the following, we assume $X \in \boldsymbol{B}$.

Since $X \in \boldsymbol{B}$, by Property 3 of RQS, $P_{3a}(Q_2, Q_c, X)$ or $P_{3b}(Q_2, Q_c, X)$ must hold. However, since $Q_2 \cap Q_c \setminus B = X$, we have $Q_2 \cap Q_c \setminus X \subseteq B$, i.e, $P_{3a}(Q_2, Q_c, X)$ does not hold.

The only step in the algorithm in which correct server s_i stores a pair c in slot 1 with some actual quorum id (i.e., when $history_i[c.ts, 1].sets$ changes the state) is when s_i receives a message sent by a reader in the writeback call in line 44, Fig. 7 (recall here that readers are benign). Since all servers from X (recall here that servers from X are correct, since $X \subseteq Q_c$) stored c in slot 1 with Q_2 (by time t), then there is at least one read rd that executes the *writeback* procedure in line 44 and sends $\mathsf{wr}\langle c.ts, c.val, \boldsymbol{Set}, 1\rangle$ (line 60), where $Q_2 \in \boldsymbol{Set}$. In this case, $\boldsymbol{BCD}(c, 2, 1) = \boldsymbol{Set}$ is not empty (since $Q_2 \in \boldsymbol{Set}$) in line 41 of rd, and the condition in line 42 is not satisfied. Since $\boldsymbol{BCD}(c, 2, 1)$ is not empty in rd, then (line 2, Fig. 7) there exists class 1 quorum Q_1 such that for every benign server s_j in $Q_1 \cap Q_2$, $history[j, c.ts, 1].pair = c$ holds in rd (notice that this holds before time t). Therefore, every benign server from $Q_1 \cap Q_2$ stored c in slot 1 before time t. Hence, if $Y = Q_1 \cap Q_2 \cap Q_c \notin \boldsymbol{B}$, the lemma follows ($Y$ is the desired basic subset). Therefore, in the following, we assume $Y \in \boldsymbol{B}$.

We have showed that (by time t), every server s_i from the set $Z = X \cup Y$ stored c in slot 1 and that $Z \subseteq Q_c$ (since $X \subseteq Q_c$ and $Y \subseteq Q_c$). We now show that Z is the desired basic subset, i.e., we show that it is not possible that $Z \in \boldsymbol{B}$.

Assume by contradiction that $Z \in \boldsymbol{B}$. Then, by Property 3 of RQS, $P_{3a}(Q_2, Q_c, Z)$ or $P_{3b}(Q_2, Q_c, Z)$ must hold. However, since (i) $P_{3a}(Q_2, Q_c, X)$ does not hold and (ii) $X \subseteq Z$, $P_{3a}(Q_2, Q_c, Z)$ cannot hold either (we have $Q_2 \cap Q_c \setminus Z \subseteq B$). Moreover, since (i) $P_{3b}(Q_2, Q_c, Y)$ does not hold (since $Q_1 \cap Q_2 \cap Q_c = Y$) and (ii) $Y \subseteq Z$, $P_{3b}(Q_2, Q_c, Z)$ cannot hold either (we have $Q_1 \cap Q_2 \cap Q_c \subseteq Z$). A contradiction. □

Theorem 8. ***(Wait-freedom.)*** *The algorithm in Figures 5, 6 and 7 is wait-free.*

Proof. The argument for the wait-freedom of a write operation is straightforward; in every round of a write, the writer waits for acks from at least one quorum, so the writer is guaranteed to receive the awaited acks eventually, since we assume existence of quorum Q_c that contains only correct servers. The timer that the writer awaits eventually expires and write eventually completes.

The argument for the wait-freedom of a read operation is more involved. We show that any read operation invoked by a correct client does not block in line 34, Fig. 7; the remainder of the proof is straightforward. We distinguish two cases: (1) the case where there is an infinite (unbounded) number of write operations in the execution, and (2) the case where the writer issues a finite number of write operations in the execution.

1. In this case, there is an infinite number of writes. Suppose, by contradiction, that rd never completes. Let ts equal $highest\_ts$ computed at the end of round 1 of rd, in line 29, Fig. 7. Since the writer issues an unbounded number of writes, the writer will also issue a write with a timestamp ts, writing some value v. Since all benign servers from some quorum Q store $\langle ts, v\rangle$ in slot 1 at some time t and, since by Property 1 of RQS $Q \cap Q_c$ is a basic subset, $safe(\langle ts, v\rangle)$ holds after rd receives at least one ack from every server from Q_c sent after t. Moreover, for all other pairs c with $c.ts > ts$, $invalid(c)$ will also hold (since $c.ts > highest\_ts = ts$) and, hence, $highCand(\langle ts, v\rangle)$ also eventually holds. Hence $\langle ts, v\rangle$ is eventually in C and rd terminates. A contradiction.
2. In this case, there is a write operation with the highest timestamp. Let wr denote the last complete write operation that writes v with timestamp ts (or $v = \perp$, $ts = 0$ if there is none). We denote by wr' a possible later (incomplete) write that writes v' with ts'.
   Assume, by contradiction, that read rd never returns a value. First consider the case, where $ts < highest\_ts$ (where $highest\_ts$ is computed in line 29, Fig 7).
   Then, rd invokes rounds on all correct servers, sending rd messages infinitely many times. We distinguish two cases: (a) there is no basic subset $T \subseteq Q_c$ such that all servers from T ever

stores $c' = \langle ts', v'\rangle$ in slot 1, and (b) there is a time t at which all servers from some basic subset $T \subseteq Q_c$ store c' in slot 1. In case (a), let t be the time at which the last correct server stores c' in slot 1. Moreover, let $t' > t$ be the time at which rd receives at least one response from every server from Q_c sent after t (in both cases (a) and (b)).

(a) Since wr completed, there is a quorum Q such that all benign servers from Q have stored $\langle ts, v\rangle$ in slot 1. By Property 1 of RQS, $Q \cap Q_c = T_v$ is a basic subset. Hence, from time t' onward, rd received at least one ack from all servers from T_v sent after wr completed and, hence, $safe(\langle ts, v\rangle)$ holds.
Moreover, by Lemma 15 and assumption (a), $V_2(c, Q_c)$ never holds for some pair c such that $c.ts > ts$. Similarly, by Lemma 16 and assumption (a), $V_3(c, Q_c)$ never holds for such a pair c. Therefore, for every timestamp-value pair c, such that $c.ts > ts$, $valid_2(c, Q_c)$ and $valid_3(c, Q_c)$ cannot hold in rd. Finally, by our assumption (a) no $T \subseteq Q_c$ stores c in slot 1, such that $c.ts > ts$. Therefore, after time t', for any value c, such that $c.ts > ts$, $valid_1(c, Q_c)$ does not hold. Hence, at the next iteration, $invalid(c)$ holds for all c such that $c.ts > ts$ and, therefore, $highCand(\langle ts, v\rangle)$ holds; moreover, since $safe(\langle ts, v\rangle)$ also holds, $\langle ts, v\rangle$ is in set C in line 33 and rd returns, a contradiction.

(b) In this case, after t', there is a basic subset $T \subseteq Q_c$ for which $history[*, ts', 1].pair = \langle ts', v'\rangle$. Hence, $safe(\langle ts', v'\rangle)$ holds after t'. It is not difficult to see, since no subsequent valid value is present in the system (since wr' is the last write invoked), that for every timestamp/value pair c'' such that $c''.ts > c'.ts \vee (c''.ts = c'.ts \wedge c''.val \neq c'.val)$ none of the predicates $valid_1(c'', Q_c)$, $valid_2(c'', Q_c)$ or $valid_3(c'', Q_c)$ holds, i.e., $invalid(c'')$ holds. Hence, $highCand(\langle ts', v'\rangle)$ also holds. Thus, in the next iteration, $\langle ts', v'\rangle \in C$ and read returns: a contradiction.

Consider now the case, where $ts \geq highest\_ts$. Since write wr (with timestamp ts) completed, then a write with a timestamp $highest\_ts$ also completed. It is not difficult to see (along the lines of the proof of case (1)) that rd returns the value written with timestamp $highest\_ts$. □

Theorem 9. *(**Best-Case Latency.**) The storage algorithm in Figures 5, 6 and 7 is* $(m, \boldsymbol{QC_m})$*–*fast *for all* $m \in \{1, 2, 3\}$.

Proof. For write operation, the proof is straightforward. For read, it is important to show that whenever the read is synchronous and uncontended, lines 20-35 are executed only once. This proof is given in the following. The rest of the proof is straightforward, by using the output of BCD (lines 1-2, Fig. 7).

Since there is no contention, let wr writing timestamp value pair $c = \langle ts, v\rangle$ be the last (complete) write that precedes the read rd. Regardless of whether wr completed in 1, 2, or 3 rounds, wr wrote $c = \langle ts, v\rangle$ into some quorum of servers Q. Moreover, no benign server stores any value with a higher timestamp than ts by Lemma 6. Since rd is synchronous, a quorum Q_c that contains only correct server will respond in the first round of rd. By Property 1 of RQS and Definition 5 $Q_c \cap Q = T_c$ is a basic subset that contains only correct servers, and, hence, $safe(c)$ holds at the end of round 1 of rd. It is not difficult to see that for any value $c'.val$ with $c'.ts > ts$, none of the predicates $valid_1(c', Q_c)$, $valid_2(c', Q_c)$ and $valid_3(c', Q_c)$ will hold. Hence, for any such timestamp/value pair $invalid(c')$ holds. Hence, at the end of round 1 of rd, $highCand(c)$ also holds and hence $c \in C$ in line 33, Fig. 7. □

B Correctness of the consensus algorithm

In this Appendix we prove the correctness of our consensus algorithm of Section 4.2. First, we give few definitions.

Definition 6 (Value decided in a view). *We say that a value v is* Decided-2, Decided-3 *or* Decided-4 *in view w, if there is a benign process (acceptor or learner) p that eventually decides a value by receiving (respectively):*

- *(Decided-2)* $\mathsf{update}_1\langle v, w, *\rangle$ *messages from a class 1 quorum (line 51, Fig. 15).*
- *(Decided-3)* $\mathsf{update}_2\langle v, w, Q_2\rangle$ *messages from a class 2 quorum Q_2 (line 52, Fig. 15).*
- *(Decided-4)* $\mathsf{update}_3\langle v, w, *\rangle$ *messages from some quorum (line 53, Fig. 15).*

We also say that a value v is decided *in view w, if some benign process p* Decided-m *v in view w (where $m \in \{2, 3, 4\}$).*

Definition 7 (Prepares). *We say that an acceptor a_i* prepares *a value v in view w, if it eventually receives* $\mathsf{prepare}\langle v, w, *, *\rangle$ *and executes lines 31-33, Fig. 15.*

Definition 8 (Updates). *We say that a benign acceptor a_i* updates *a value v in view w, if it eventually receives* $\mathsf{update}_{step}\langle v, w, *\rangle$ *for some $step \in \{1, 2\}$ and executes lines 34-38, Fig. 15. More precisely, we say a_i* 1-updates *(resp.,* 2-updates*) v in w if $step = 1$ (resp., $step = 2$).*

Definition 9 (Accepts). *We say that a benign acceptor a_i* accepts *a value v in view w, if it prepares or updates v in view w.*

We also make use of the Definition 5 of Appendix A (definition of *basic* and *large* subsets).
In addition, we introduce the following notation:

- We say that an invocation of function $choose(*, vProof, Q)$ is *valid* if $vProof$ consists of valid new_view_ack messages sent by acceptors from quorum Q (with slight abuse of language, we also simply say $choose(*, vProof, Q)$ is *valid*);
- We denote all Byzantine acceptors in execution ex by B_{ex}. We assume $B_{ex} \in \boldsymbol{B}$ for any execution ex.

Lemma 17. ***Size of basic sets.*** *In every execution of our consensus algorithm, any basic subset contains at least one benign acceptor.*

Proof. The lemma follows directly from the definition of a basic subset (Definition 5). □

Lemma 18. ***Size of large sets.*** *In every execution ex of our consensus algorithm. for any large subset T_2, there is a basic subset $T_1 \subseteq T_2$ that contains only benign acceptors.*

Proof. By Definition 5, for any large subset T_2, $T_2 \setminus B_{ex}$ is a basic subset. By definition of B_{ex}, $T_2 \setminus B_{ex}$ contains only benign acceptors. Hence the lemma. □

We first prove *Validity.*

Lemma 19. ***Validity of the choose function.*** *If valid $choose(v, vProof, Q)$ returns v such that v is a candidate with view w, then at least one benign acceptor a_i prepared v in w.*

Proof. Assume $Cand_2(v, w, Q)$ holds (line 1, Fig. 13). In this case, every acceptor a_j from the set $X = (Q_1 \cap Q) \setminus B$ (where B is not a basic subset and Q_1 is a class 1 quorum) reported that it prepared v in w. Note that, by Property 2 of RQS, $Q_1 \cap Q$ is a large subset. By Lemma 18, X contains at least one benign acceptor.

Assume now $Cand_3(v, w, char, Q)$ holds (for $char \in \{\text{`}a\text{`}, \text{`}b\text{`}\}$). From lines 2-3, Fig. 13, it follows that all acceptors from the set $X = (Q_2 \cap Q) \setminus B$, (where B is not a basic subset and Q_2 is a class 2 quorum) reported that they updated v in w (i.e., $\forall a_j \in X : vProof[a_j].Update[1] = v$ and $w \in vProof[a_j].Update_{view}[1]$). Note that, by Property 1 of RQS, $Q_2 \cap Q$ is a basic subset. Hence, by Definition 5, X is a non-empty set. In this case, $vProof[a_j].Update_{proof}[1, w]$ contains at least a basic subset of signed $\mathsf{update}_1\langle v, w, *\rangle$ messages. By Lemma 17 at least one of these signed messages comes from a benign acceptor a_i that indeed prepared v in view w.

The argument for the case where $Cand_4(v, w, Q)$ holds (line 5, Fig. 13) is very similar to the case where predicate $Cand_3(v, w, char, Q)$ holds. □

Theorem 10. ***(Validity)*** *If a benign learner learns a value v and all proposers are benign, then some proposer proposed v.*

Proof. A benign learner learns a value v by receiving (1) update_* messages (lines 51-53 and 60, Fig. 15), or (2) by receiving a basic subset of decision messages (line 101, Fig. 15). In case (b), by Lemma 17 and line 40, Fig. 15 at least one benign acceptor decided v before the learner learned v.

Hence, in both cases, if a learner learns v, then v was accepted in some view w (prepared or updated) by benign acceptors from some quorum of acceptors. Since any quorum is a basic subset, there is at least one such benign acceptor a_j (by Property 1 of RQS, and Lemma 17). Note that a_j updates v in w only upon a_j prepares v in w. We prove the following statement using induction on view numbers: *if a benign acceptor prepares v in view w, then some proposer proposed v.*
Base Step: $(w = initView)$

Benign acceptors prepare value v in $initView$ only if they receive a $\mathsf{prepare}\langle v, initView, *, *\rangle$ message from some proposer. Since all proposers are benign, no proposer sends a prepare message containing v unless it proposes v. Hence, if some benign acceptor accepts v, v was indeed proposed by some proposer.

Inductive Hypothesis (IH): For every view $w, w' > w \geq initView$, if a benign acceptor accepts v in w, then some proposer proposed v.

Inductive Step: We prove that the statement is true for view w'. In view w', benign acceptors accept only values returned by valid $choose(*, vProof, Q)$. If $choose(*, vProof, Q)$ returns a candidate value v, by Lemma 19, some benign acceptor prepared v in view $w, w < w'$, and by IH, v was proposed by some proposer. If $choose(*, vProof, Q)$ returns in line 21, Figure 13, then the returned value v is the initial proposal value of the leader of w'. We conclude that v was proposed by some proposer. □

Now we prove *Agreement.*

Lemma 20. *After sending a new_view_ack message for view w, a benign acceptor cannot accept a value v with view number $w' < w$.*

Proof. By line 21, Fig. 15 a benign acceptor cannot prepare a value with $w' < w$. Moreover, a benign acceptor a_i updates a value v in some view w'' only after a_j prepares v in w''. Hence the lemma. □

Lemma 21. *If two values v and v' are decided in view w, then $v = v'$.*

Proof. Suppose $v \neq v'$. From Def. 6, all acceptors from some quorum Q (resp., Q') sent $\mathsf{update}_m\langle v, w\rangle$ (resp., $\mathsf{update}_{m'}\langle v', w\rangle$) message, for some $m, m' \in \{1, 2, 3\}$. Hence, all benign acceptors from Q (resp., Q') prepare v (resp., v') in w. By Property 1 of RQS and Definition 5, $Q \cap Q'$ is a basic subset, which contains at least one benign acceptor a_i (by Lemma 17). That is, there exists a benign acceptor that prepared different values in the same view. A contradiction. □

Lemma 22. ***Unique Cand2(v,w,Q).*** *There are no two different values v and v' such that, in valid $choose(*, vProof, Q)$, both $Cand_2(v', w, Q)$ and $Cand_2(v, w, Q)$ hold, for the same w.*

Proof. Assume by contradiction that such values v and v' exist. By definition of the predicate $Cand_2()$ (line 1, Fig. 13), there are sets $X = (Q_1 \cap Q) \setminus B$ and $X' = (Q_1' \cap Q) \setminus B'$, such that (1) $B, B' \subseteq acceptors$ and B and B' are not basic subsets, (2) Q_1 and Q_1' are class 1 quorums, and (3) all acceptors from the set X (resp., X') prepared v (resp., v') in w. By Property 2 of RQS, $Q_1 \cap Q_1' \cap Q$ is a large subset. Applying Definition 5 we conclude that $X \cap X'$ is a non-empty set. Hence, there is an acceptor $a_j \in Q$ such that $vProof[a_j].Prep = v$ and $vProof[a_j].Prep = v'$. Hence, $v = v'$. A contradiction. □

Lemma 23. ***Cand3(v,w,'a',Q)/Cand4(v,w,Q).*** *If for some value v $Cand_3(v, w, \text{`}a\text{'}, Q)$ or $Cand_4(v, w, Q)$ hold in valid $choose(*, vProof, Q)$, then all benign acceptors from some quorum Q' prepared v in w.*

Proof. Assume first $Cand_3(v, w, \text{`}a\text{'}, Q)$ holds. Then there is a set $X = (Q_2 \cap Q) \setminus B'$ such that B' is not a basic subset and Q_2 is a class 2 quorum and $P_{3a}(Q_2, Q, B')$ holds. By Property 3a of RQS, X is a basic subset. By Lemma 17 there is at least one benign acceptor in X that updated v in w. Therefore, by lines 34-38 in Fig. 15, all benign acceptors from some quorum Q' prepared v in w.

Assume now $Cand_4(v, w, Q)$ holds. Then there exists an acceptor $a_j \in Q$ such that:

- $vProof[a_j].Update[2] = v$,
- $w \in vProof[a_j].Update_{view}[2]$, and
- $vProof[a_j].Update_{proof}[2, w]$ contains a basic subset of signatures of $\mathsf{update}_2\langle v, w, *\rangle$ messages including at least one from a benign acceptor a_b.

Hence a benign acceptor a_b updated v in w. Therefore, all benign acceptors from some quorum Q' prepared v in w. □

Lemma 24. ***Impossible candidates after decision.*** *If value v is decided in some view w, then, in any valid $choose(*, vProof, Q)$, for some $v' \neq v$, neither $Cand_3(v', w, \text{`}a\text{'}, Q)$ nor $Cand_4(v', w, Q)$ can hold.*

Proof. In case v was decided in view w, by Definitions 6, 7 and 8, all benign acceptors from a quorum Q prepared v in w.

Assume, by contradiction, that such value $v' \neq v$ exists, such that $Cand_3(v', w, \text{`}a\text{'}, Q)$ or $Cand_4(v', w, Q)$ hold and $v' \neq v$. Then, by Lemma 23, all benign acceptors from some quorum Q' prepared v' in w. By Property 1 of RQS, $Y = Q \cap Q'$ is a basic subset that contains at least one benign acceptor which prepared both v and v' in w. A contradiction. □

Lemma 25. *If w is the lowest view number in which some value v is Decided-2, then no benign acceptor a_i prepares any value v', $v' \neq v$ in any view higher than w.*

Proof. We prove this lemma by induction on view numbers.
Base Step: First, we prove the lemma for view $w+1$. A benign acceptor a_i prepares a value v' in some view $W > w$ only if the valid $choose(*, vProof, Q)$ function in view W returns v', without setting the *abort* flag. Therefore, it is sufficient to prove that for a valid $choose(*, vProof, Q)$ in view $w+1$ returns v, or *abort* flag is set.

By Definitions 6 and 7, all benign acceptors from a class 1 quorum Q_1 prepared v in w. By definition of B_{ex}, set $X = (Q_1 \cap Q) \setminus B_{ex}$ that contains only benign acceptors. By Lemma 20, every acceptor $a_i \in X$ prepared v in w, before replying with the new_view_ack message to the leader of the view $w+1$. In the meantime, no acceptor $a_j \in X$ prepared any other value, as this would mean that a_j would be in a higher view than $w+1$ when replying with new_view_ack for the view $w+1$, which is impossible. Therefore, $Cand_2(v, w, Q)$ (line 1, Fig. 13) holds in $choose(*, vProof, Q)$, for any Q. Notice that for every acceptor $a_j \in X$, $w \in vProof[a_j].Prep_{view}$.

By Lemma 19, it is not difficult to see that there is no value v' such that $Cand_2(v', w', Q)$, $Cand_3(v', w', *, Q)$ or $Cand_4(v', w', Q)$ holds for some $w' > w$.

By Lemma 22, there is no value $v' \neq v$ such that $Cand_2(v', w, Q)$ holds.

By Lemma 24, there is no value $v' \neq v$ such that (i) $Cand_3(v', w, \text{`}a\text{`}, Q)$ holds or (ii) $Cand_4(v', w, Q)$ holds.

Finally, in the following part of the proof, we show that it is not possible that both $Cand_3(v', w, \text{`}b\text{`}, Q)$ and $Valid_3(v', w, \text{`}b\text{`}, Q)$ hold for $v' \neq v$.

Assume, by contradiction, that there is such a value $v' \neq v$ such that both $Cand_3(v', w, \text{`}b\text{`}, Q)$ and $Valid_3(v', w, \text{`}b\text{`}, Q)$. Since $Cand_3(v', w, \text{`}b\text{`}, Q)$ holds, there are class 2 quorum Q_2 and $B \in \boldsymbol{B}$ such that $C_3(v', w, \text{`}b\text{`}, Q_2, B, Q)$ holds (line 2, Fig. 13). Moreover, since $Valid_3(v', w, \text{`}b\text{`}, Q)$ holds, all acceptors from $Y = Q_2 \cap Q$ claim they prepared v' in w (line 2, Fig. 13) or do not have w in $vProof[a_j].Prep_{view}$. Since we know that for all (benign) servers from $a_j \in X = (Q_1 \cap Q) \setminus B_{ex}$, $v \in vProof[a_j].Prep$ and $w \in vProof[a_j].Prep_{view}$, we conclude $X \cap Y = \emptyset$. Hence, we have $Q_1 \cap Q_2 \cap Q \setminus B_{ex} = \emptyset$, i.e., $P_{3b}(Q_2, Q, B_{ex})$ does not hold. By Property 3 of RQS, $P_{3a}(Q_2, Q, B_{ex})$ must hold.

Moreover, since $C_3(v', w, \text{`}b\text{`}, Q_2, B, Q)$, all acceptors from $Z = Q_2 \cap Q \setminus B$ claim that all acceptors from Q_2 prepared v' in w. We distinguish two cases: (1) all acceptors from Z are Byzantine, i.e., $Z \subseteq B_{ex}$, and (2) there is a benign server in Z.

In case (1), since $Z \subseteq B_{ex}$ and $P_{3a}(Q_2, Q, B_{ex})$ holds, we have $P_{3a}(Q_2, Q, Z)$. Hence $Q_2 \cap Q \setminus Z \notin \boldsymbol{B}$. However, by definition of Z, $Q_2 \cap Q \setminus Z \subseteq B \in \boldsymbol{B}$. A contradiction.

In case (2), since Z contains at least one benign acceptor s_i, then all benign acceptors from Q_2 prepared v' in w. By Property 2 of RQS, $Q_1 \cap Q_2 \setminus B_{ex}$ is a basic subset that contains only benign acceptors, that all prepared both v and v' in w. A contradiction.

By inspection of $choose()$ pseudocode, $choose()$ returns v or *abort* flag is set.

Inductive Hypothesis (IH): Assume that no benign acceptor a_i prepares any value different from v in any view from $w+1$ to $w+k$. We prove that no benign acceptor a_i can prepare any value different from v in the view $w+k+1$.

Inductive Step: It is sufficient to prove that for a valid $choose(*, vProof, Q)$ in view $w+k+1$ returns v, or *abort* flag is set.

By Definitions 6 and 7, all benign acceptors from a class 1 quorum Q_1 prepared v in w. By IH, all benign acceptors from Q_1 can prepare only v in views $w+1$ to $w+k$. Set $X = (Q_1 \cap Q) \setminus B$ contains only benign acceptors; by Lemma 20, no acceptor $a_i \in X$ prepares a value in a higher view than $w+k$ before sending a new_view_ack message to the leader of the view $w+k+1$. Hence, by definition of predicate $Cand_2()$, $Cand_2(v, w, Q)$ holds in $choose(*, vProof, Q)$, for any Q. Notice that for every acceptor $a_j \in X$, $w \in vProof[a_j].Prep_{view}$.

By Lemma 22, there is no other value $v' \neq v$ such that $Cand_2(v', w, Q)$ holds.

By Lemma 19 and IH, it is not difficult to see that there is no value $v' \neq v$ such that $Cand_2(v', w', Q)$, $Cand_3(v', w', *, Q)$ or $Cand_4(v', w', Q)$ holds for some $w' > w$.

By Lemma 24, there is no value $v' \neq v$ such that $Cand_3(v', w, \text{‘}a\text{‘}, Q)$ holds or $Cand_4(v', w, Q)$ holds.

Finally, exactly as in the Base Step, it can be shown that it is not possible that both $Cand_3(v', w, \text{‘}b\text{‘}, Q)$ and $Valid_3(v', w, \text{‘}b\text{‘}, Q)$ hold for $v' \neq v$.

By inspection of $choose()$ pseudocode, $choose()$ returns v or *abort* flag is set. □

Similarly to Lemma 25, we prove the following two lemmas using the properties of RQS and induction on view numbers.

Lemma 26. *If w is the lowest view number in which some value v is Decided-3, then no benign acceptor a_i prepares any value v', $v' \neq v$ in any view higher than w.*

Proof. We prove this lemma by induction on view numbers.
Base Step: First, we prove the lemma for view $w+1$. It is sufficient to prove that any valid $choose(*, vProof, Q)$ in view $w+1$ returns v, or *abort* flag is set.

By Definitions 6 and 8, all benign acceptors from a class 2 quorum Q_2 updated-1 (and prepared) v in w. By definition of B_{ex}, set $X = (Q_2 \cap Q) \setminus B_{ex}$ contains only benign acceptors. By Lemma 20, every acceptor $a_i \in X$ prepared v in w, before replying with the new_view_ack message to the leader of the view $w+1$. In the meantime, no acceptor $a_j \in X$ prepared any other value, as this would mean that a_j would be in the higher view then $w+1$ when replying with new_view_ack for the view $w+1$, which is impossible. Therefore, for some $char \in \{\text{‘}a\text{‘}, \text{‘}b\text{‘}\}$, $C_3(v, w, char, Q_2, B_{ex}, Q)$ and $Cand_3(v, w, char, Q)$ (lines 2-3, Fig. 13) hold in $choose(*, vProof, Q)$, for any Q.

By Lemma 19, it is not difficult to see that there is no value v' such that $Cand_2(v', w', Q)$, $Cand_3(v', w', *, Q)$ or $Cand_4(v', w', Q)$ holds for some $w' > w$.

By Lemma 24, there is no value $v' \neq v$ such that $Cand_3(v', w, \text{‘}a\text{‘}, Q)$ holds or $Cand_4(v', w, Q)$ holds.

We distinguish two cases: (a) $Cand_3(v', w', \text{‘}a\text{‘}, Q)$, and (b) $Cand_3(v', w', \text{‘}b\text{‘}, Q)$ holds. By inspection of $choose()$ pseudocode, in case (a) $choose()$ returns v, whereas in case (b) either $choose()$ returns v or *abort* flag is set.

Inductive Hypothesis (IH): Assume that no benign acceptor a_i prepares any value different from v in any view from $w+1$ to $w+k$. We prove that no benign acceptor a_i can prepare any value different from v in the view $w+k+1$.

Inductive Step: It is sufficient to prove that any valid $choose(*, vProof, Q)$ in view $w+k+1$ returns v, or *abort* flag is set.

By Definitions 6 and 8, all benign acceptors from a class 2 quorum Q_2 update-2 v in w. By IH, all benign acceptors from Q_2 can update-2 only v in views $w+1$ to $w+k$. All acceptors from $X = (Q_2 \cap Q) \setminus B_{ex}$ are benign and, by Lemma 20, no acceptor $a_i \in X$ prepares (nor 1-updates) a value in a higher view than $w+k$ before sending a new_view_ack message to the leader of the view $w+k+1$. Hence, for every $a_i \in X$ $vProof[a_i].Update[1] = v$ and $w \in vProof[a_i].Update_{view}[1]$. Hence, by definition of predicate $Cand_3()$, for some $char \in \{‘a‘, ‘b‘\}$, $C_3(v, w, char, Q_2, B_{ex}, Q)$ and $Cand_3(v, w, char, Q)$ hold in $choose(*, vProof, Q)$, for any Q.

By Lemma 19 and IH, it is not difficult to see that there is no value $v' \neq v$ such that $Cand_2(v', w', Q)$, $Cand_3(v', w', *, Q)$ or $Cand_4(v', w', Q)$ holds for some $w' > w$.

By Lemma 24, there is no value $v' \neq v$ such that $Cand_3(v', w, ‘a‘, Q)$ holds or $Cand_4(v', w, Q)$ holds.

We distinguish two cases: (a) $Cand_3(v', w', ‘a‘, Q)$, and (b) $Cand_3(v', w', ‘b‘, Q)$ holds. By inspection of *choose*() pseudocode, in case (a) *choose*() returns v, whereas in case (b) it either returns v or *abort* flag is set. □

Lemma 27. *If w is the lowest view number in which some value v is Decided-4, then no benign acceptor a_i prepares any value v', $v' \neq v$ in any view higher than w.*

Proof. We prove this lemma by induction on view numbers.
Base Step: First, we prove the lemma for view $w+1$. It is sufficient to prove that any valid $choose(*, vProof, Q)$ in view $w+1$ returns v.

By Definitions 6 and 8, all benign acceptors from some quorum Q_3 updated-2 v in w. By Property 1 of RQS $X = Q_3 \cap Q$ is a basic subset that contains at least one benign acceptor a_i (by Lemma 17). By Lemma 20, a_i updated-2 v in w, before replying with the new_view_ack message to the leader of the view $w+1$. In the meantime, a_j did not prepare (nor update-2) any other value, as this would mean that a_j would be in the higher view then $w+1$ when replying with new_view_ack for the view $w+1$, which is impossible. Therefore, $Cand_4(v, w, Q)$ (line 5, Fig. 13) holds in $choose(*, vProof, Q)$, for any Q.

By Lemma 19, it is not difficult to see that there is no value v' such that $Cand_2(v', w', Q)$, $Cand_3(v', w', *, Q)$ or $Cand_4(v', w', Q)$ holds for some $w' > w$.

By Lemma 24, there is no value $v' \neq v$ such that $Cand_3(v', w, ‘a‘, Q)$ holds or $Cand_4(v', w, Q)$ holds.

By inspection of *choose*() pseudocode, *choose*() returns v.

Inductive Hypothesis (IH): Assume that no benign acceptor a_i prepares any value different from v in any view from $w+1$ to $w+k$. We prove that no benign acceptor a_i can prepare any value different from v in the view $w+k+1$.

Inductive Step: It is sufficient to prove that any valid $choose(*, vProof, Q)$ in view $w+k+1$ returns v.

By Definitions 6 and 8, all benign acceptors from some quorum Q_3 updated-2 v in w. By IH, all benign acceptors from Q_3 can prepare (and, hence, update-2) only v in views $w+1$ to $w+k$. Moreover, $X = Q_3 \cap Q$ contains at least one benign acceptor a_j (by Property 1 of RQS and Lemma 17). Hence, by definition of predicate $Cand_4()$, $Cand_4(v, w', Q)$ holds in $choose(*, vProof, Q)$, for any Q, for some $w', w+k \geq w' \geq w$.

By Lemma 19 and IH, it is not difficult to see that there is no value $v' \neq v$ such that $Cand_2(v', w'', Q)$, $Cand_3(v', w'', *, Q)$ or $Cand_4(v', w'', Q)$ holds for some $w'' > w$.

By Lemma 24, there is no value $v' \neq v$ such that $Cand_3(v', w, `a`, Q)$ holds or $Cand_4(v', w, Q)$ holds.

By inspection of *choose*() pseudocode, *choose*() returns v. □

Theorem 11. ***(Agreement)*** *No two benign learners learn different values.*

Proof. If a benign learner learns a value then a value was decided in some view (by some benign process). Indeed, a benign learner learns a value v by receiving (1) update$_*$ messages (lines 51-53, and 60 Fig. 15), or (2) by receiving a basic subset of decision messages (line 101, Fig. 15). In case (b), by Lemma 17 at least one benign acceptor decided v before the learner learned v.

It is not difficult to see that, if some value v' is decided in view w, then some benign acceptor prepared v' in w. The theorem follows from Lemmas 21, 25, 26 and 27. □

It is straightforward to show that our algorithm is $(m, \boldsymbol{QC_m})$*–fast*, for $m \in \{1, 2, 3\}$.

The following two lemmas are critical for ensuring *Termination* property.

The first lemma proves that our algorithm does not block in lines 3-8, Fig. 15, in case some quorum contains only correct acceptors.

Lemma 28. *The abort flag is never set in valid choose*$(*, vProof, Q)$.

Proof. It is sufficient to prove that if $choose(*, vProof, Q)$ sets *abort* flag, then Q contains at least one Byzantine acceptor. We consider two exhaustive cases.

Case (a): *choose*() aborts in line 16, as there are two values v and $v' \neq v$ such that both $Cand_3(v, w, `b`, Q)$ and $Cand_3(v', w, `b`, Q)$ hold (for $w = view_{max}$). In this case, by definition of predicate $Cand_3()$ (lines 2-3, Fig. 13) there are acceptors $a_i, a_j \in Q$ and class 2 quorums Q_2 and Q_2' such that: (1) a_i claims that all (benign) acceptors from Q_2 prepared v in w, (2) a_j claims that all (benign) acceptors from Q_2' prepared v' in w. By Property 1 of RQS $Q_2 \cap Q_2'$ is a basic subset, that by Lemma 17 contains at least one benign acceptor a_x. Hence, a_i claims that a benign acceptor a_x prepared v in w, while a_j claims that a_x prepared $v' \neq v$ in w. Hence, at least one acceptor from the set $\{a_j, a_i\} \subset Q$ is Byzantine.

Case (b): *choose*() aborts in line 18, as $Cand_3(v, w, `b`, Q)$ holds but $Valid_3(v, w, `b`, Q)$ does not hold. Assume, by contradiction, that Q contains only benign acceptors. Notice that, if a benign acceptor $a_j \in Q$ prepares v in view w, then the following predicate P (extracted from definition of $Valid_3(v, w, `b`, Q)$, line 4, Fig. 13)

$$((vProof[a_j].Prep = v) \wedge (w \in vProof[a_j].Prep_{view})) \vee (w' \in vProof[a_j].Prep_{view} \Rightarrow w' > w)$$

must hold in any valid $choose(*, vProof, Q)$ for view higher than w. To see this, consider lines 31-33, Fig. 15, and notice that when benign acceptor a_j prepares v in w, $Prep = v$ and $w \in Prep_{view}$ (at a_j). If this ceases to hold at some point, then a_j prepared a different value in a view higher than w, and $Prep_{view}$ contains only view numbers higher than w (line 32, Fig. 15).

Since $Cand_3(v, w, `b`, Q)$ holds, but $Valid_3(v, w, `b`, Q)$ does not hold, there is class 2 quorum Q_2 and $B \in \boldsymbol{B}$, such that $C_3(v, w, `b`, Q_2, B, Q)$ holds, i.e., all acceptors in $X = Q_2 \cap Q \setminus B$ claim that all acceptors from Q_2 prepared v in view w. However, there is an acceptor a_i in $Q_2 \cap Q$ (since $Valid_3(v, w, `b`, Q)$ does not hold) for which the above predicate P does not hold, i.e., a_i never prepared v in w. Obviously, some acceptor from $a_i \cup X \subseteq Q$ is Byzantine. A contradiction. □

The second lemma proves that our algorithm does not block in lines 23-27, Fig. 15, in case some quorum contains only correct acceptors. In the following proof, we explicitly make use of the assumption of an eventually synchronous system, i.e., of an existence of a global stabilization time GST after which the system is synchronous.

Lemma 29. ***(Availability of signatures.)*** *If a correct acceptor a_j issues a* sign_req$\langle Update[step], w, step\rangle$ *message (line 24, Fig. 15) after GST, then a_j eventually receives signed* update$_{step}\langle Update[step], w, *\rangle$ *messages from some basic subset of acceptors.*

Proof. A correct acceptor a_j issues a sign_req$\langle Update[step], w, step\rangle$, only if (at a_j) $w \in Update_{view}[step]$, i.e., only if a_j updated a value $v = Update[step]$ in w. In other words, before issuing a sign_req message, a_j received update$_{step}\langle v, w, *\rangle$ messages from some quorum Q and executed lines 34-38, Fig. 15. In particular a_j adds the identifier of the quorum Q to the $Update_Q[step, w]$ set (line 37, Fig. 15). Without loss of generality, we can assume that a_j sent a sign_req$\langle v, w, step\rangle$ message to acceptors from quorum Q.

Let Q_c be the quorum that contains only correct acceptors. By Property 1 of RQS, $T = Q \cap Q_c$ is a basic subset. Since $T \subseteq Q_c$, T contains only correct acceptors. Since after GST the system is synchronous, a_j eventually receives the desired set of signatures. □

We also need the following two simple lemmas. In the remainder of the paper, we denote by Q_c the quorum that contains only correct acceptors.

Lemma 30. *If some process receives* decision *messages with the same value v from some quorum of acceptors Q, then every correct learner learns a value.*

Proof. Suppose, by contradiction, that some correct learner l_k never learns a value.

Let Q_c be the quorum that contains only correct acceptors. By Property 1 of RQS and assumption on Q_c, $T = Q_c \cup Q$ is a basic subset of correct acceptors that decided a value v. Denote by t the time after which all acceptors from T have decided v. By lines 102-103, Fig. 15, and our assumption that l_k never learns the value, l_k sends an infinite number of decision_pull messages to all acceptors. Those messages sent after $max(t, GST)$ are received by all acceptors from T who send decision messages to l_k. These messages are received by l_k and, by line 101 Fig. 15, l_k learns v — a contradiction. □

Towards proving *Termination*, we first show that our algorithm (or, more precisely, its *Locking* module) satisfies a weaker property we call *Eventual Obstruction-Free Termination* (EOFT), defined as follows.

Definition 10. ***(Eventual Obstruction-Free Termination.)*** *Assume a correct proposer p_k proposes a value at time t_p, after GST ($t_p > GST$) with the view number $view_{high}$ such that: (a) p_k is the leader of $view_{high}$, (b) p_k has a valid $viewProof$ for $view_{high}$, (c) no value with a view number higher than $view_{high}$ is proposed up to time t, and (d) no proposer proposes a value (with a valid $viewProof$) for the view higher than $view_{high}$ by $t_p + D_{OF}$, where $D_{OF} = 7\Delta$. Then, every correct learner eventually learns a value.*

Lemma 31. ***(EOFT.)*** *The* Locking *module of our consensus algorithm satisfies Eventual Obstruction-Free Termination property.*

Proof. The following proof relies on our assumption that, after GST, a correct acceptor takes any step in negligible time and that every message by a correct process p to a correct process q is received within Δ.

By our assumption of an eventually synchronous system, all acceptors from Q_c receive the new_view message for $view_{high}$ sent by p_k. Moreover, by assumptions (a)-(d) of Definition 10 we conclude that the condition in line 21 (Fig. 15) is satisfied for every acceptor from Q_c, which then proceeds to execute lines 22-28. By Lemma 29, this part of the code is non-blocking. In case some acceptor from Q_c sends some sign_req message (line 24), it will do so by $t_p + \Delta$, and similarly send sign_ack messages (line 29) by $t_p + 2\Delta$. Hence, all acceptors from Q_c execute line 28 of Fig. 15 and send the new_view_ack messages to p_k by $t_p + 3\Delta$. Denote the set of these new_view_ack messages sent by quorum Q_c (and received by the proposer p_k) by $vProof$. By Lemma 28, the $choose(*, vProof, Q_c)$ does not abort, but rather returns some value v. Therefore, at latest by $t_p + 4\Delta$, p_k sends the $\mathsf{prepare}\langle v, view_{high}, vProof, Q_c\rangle$ message to all acceptors — hence, all acceptors from Q_c receive this message. By assumptions (a) and (d) of Definition 10, we conclude that the condition in line 31, Fig. 15 is satisfied and that all acceptors from Q_c prepare v in $view_{high}$ and send the $\mathsf{update}_1\langle v, view_{high}, \emptyset\rangle$ message to all acceptors (and learners) by $t_p + 5\Delta$. By assumption (d) of Definition 10, given that all acceptors from Q_c prepare v in $view_{high}$, we conclude that all acceptors from Q_c send an $\mathsf{update}_2\langle v, view_{high}, Q_c\rangle$ (by $t_p + 6\Delta$), and, later, an $\mathsf{update}_3\langle v, view_{high}, Q_c\rangle$ (by $t_p + 7\Delta = t_p + D_{OF}$) to every acceptor and learner. As soon as a correct learner receives a $\mathsf{update}_3\langle v, view_{high}, Q_c\rangle$ message from every acceptor of Q_c it learns a value (lines 53 and 101, fig. 15), unless it already learned a value. ☐

We are now ready to prove *Termination*. Basically, what is left to show is that, in every execution in which a correct proposer proposes a value, eventually, the *Election* module ensures that the assumptions of Definition 10 eventually hold.

Theorem 12. *(Termination) If a correct proposer proposes a value, then eventually, every correct learner learns a value.*

Proof. Suppose, by contradiction, that some correct learner l_k never learns a value even if some correct proposer, say p_k, proposes a value.

Note that, if p_k proposes a value, by (1) lines 0 and 101-103, Fig. 14 and (2) the assumption of an eventually synchronous system, either (a) all correct acceptors eventually trigger their $suspectTimeout$ (line 0, Fig. 14), or (b) p_k receives a decision message from some quorum Q of acceptors and halts (line 104, Fig. 14). In the latter case (case (b)), by Lemma 30, every correct learner eventually learns a value — a contradiction.

We now focus on the case (a), where all correct acceptors eventually trigger $suspectTimeout$ (line 0, Fig. 14) – we denote this time by $t_{trigger}$. Let $GST' = max(GST, t_{trigger})$.

We distinguish two sub-cases: (i) when no correct acceptor stops its $suspectTimeout$ permanently (i.e., no correct acceptor executes the line 7, Fig. 14), and (ii) when some correct acceptor stops its $suspectTimeout$ permanently.

We first consider case (i). We define functions $view_{min}(t) = min\{nextView_{a_i} | a_i \in Q_c\}$ at time t, and, similarly, $view_{max}(t) = max\{nextView_{a_i} | a_i \in Q_c\}$. It is not difficult to see (lines 1-5, Fig. 14), that, in case (i), at every correct acceptor a_j, variable $nextView_{a_j}$ is: (1) monotonically increasing, (2) unbounded, and (3) non-skipping (it always increments by one). Hence, every correct acceptor sends an infinite number of view_change messages, for every view number from 1 (i.e., $initView + 1$) to ∞. Moreover, functions $view_{min}(t)$ and $view_{max}(t)$ are also monotonically increasing and unbounded.

Let $viewGST' = view_{max}(GST')$. Hence, every correct proposer receives all view_change messages sent by acceptors from Q_c for view numbers $viewGST' + 1$ and higher. Therefore, every correct proposer p_k proposes a value in every view $w_k \geq viewGST' + 1$, such that $k = (w_k \mod |proposers|)$.

Note that a correct acceptor a_i, on sending a view_change message with a view number w, at some time t_w, triggers a timer equal to $initTimeout * 2^w$ (lines 1-5, Fig. 14). Upon expiration of this timeout, a_i sends the subsequent view_change message. Hence, the time between a_i sends the view_change messages for view numbers w and $w + 1$ is at most $initTimeout * 2^w$.

Let t be any point time in time after GST'. Let $view(t)$ be the first view in which p_k proposes a value, such that $view(t) > view_{max}(t) + 1$. Note that no acceptor from Q_c sends a view_change message for a view higher than $view(t)$ before $T_{OF}(t) = t + initTimeout * 2^{view(t)}$. By Property 1 of RQS and the proposer code of an *Election* module, we conclude that no proposer can propose a value with a valid $viewProof$ and the view number higher than $view(t)$ before $T_{OF}(t)$.

On the other hand, all acceptors from the quorum Q_c will send the view_change message for the $view(t)$ at latest by $T_{vc}(t) = t + InitTimeout * (2^{view_{min}(t)} + 2^{view_{min}(t)+1} + \ldots + 2^{view(t)-1})$. These will be received by p_k, which will propose a value with a view number $view(t)$ at latest by $T_{prop}(t) = T_{vc}(t) + \Delta$.

Therefore, p_k proposes a value with $view(t)$ at $T_{prop}(t) > GST$, and (a) p_k is the leader of $view(t)$, (b) p_k has a valid $viewProof$ for $view(t)$ (view_change messages from Q_c), (c) no value with a view number higher than $view(t)$ is proposed up to time $T_{prop}(t)$, and (d) no proposer proposes a value (with a valid $viewProof$) for the view higher than $view(t)$ by $T_{OF}(t)$. Hence, in order to apply Lemma 31 and reach contradiction, we need to show that there exist t', such that $T_{OF}(t') - T_{prop}(t') > D_{OF}$ (where $D_{OF} = 7\Delta$).

Since $T_{OF}(t') - T_{prop}(t') = initTimeout * (2^{view(t')} - (2^{view_{min}(t')} + 2^{view_{min}(t')+1} + \ldots + 2^{view(t')-1})) - \Delta$, $T_{OF}(t') - T_{prop}(t') > D_{OF} \Leftrightarrow 2^{view_{min}(t')-1} > (D_{OF} + \Delta)/initTimeout = c$, where c is a constant. Since $view_{min}(t)$ is monotonically increasing and unbounded, such t' exists.

In case (ii) the contradiction follows directly from Lemma 30. □

C Errata

There was an omission in the conference version of this paper [22], related to the proofs of optimality of atomic storage and consensus algorithms. This caused an error in the statement of Property 3 of RQS.

In the context of Property 3, Section 2.1, the definition in [22] stated that, for a given class 2 quorum Q_2 and quorum Q, $P_{3a}(Q_2, Q, B)$ holds for all $B \in \boldsymbol{B}$, or $P_{3b}(Q_2, Q, B)$ holds for all $B \in \boldsymbol{B}$. Consequently, the algorithms presented in [22], stated as optimal, are actually not. On the other hand, algorithms presented in this paper are optimal.

We would like to thank the anonymous reviewers for pointing out the above mentioned omission, which allowed us to correct the mistakes from [22].